CHARACTERIZATION OF MODEL SETS BY DYNAMICAL SYSTEMS

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Abstract. It is shown how regular model sets can be characterized in terms of regularity properties of their associated dynamical systems. The proof proceeds in two steps. First, we characterize regular model sets in terms of a certain map β and then relate the properties of β to ones of the underlying dynamical system. As a by-product, we can show that regular model sets are, in a suitable sense, as close to periodic sets as possible among repetitive aperiodic sets.

1. Introduction

Delone sets provide an important model class for the description of aperiodic order. In particular, they can be viewed as a mathematical abstraction of the set of atomic positions of a physical quasicrystal (at zero temperature, or at a given instant of time). Many of the rather intriguing spectral properties of quasicrystals can be formulated, in a simplified manner, on the basis of Delone sets. The latter contain the important class of model sets (see below for definitions), which is our main topic in this paper.

Since the discovery of quasicrystals [34], model sets have been a particular focus of attention because they are, except under extreme conditions, pure point diffractive [14, 24, 36]. This property remains true also under certain equivariant perturbations, which turn them into deformed model sets [14, 8, 3], and extend the applicability of these sets considerably [37].

Model sets are discrete point sets that arise by (partial) projection of a lattice from some "higher dimensional" or "super" space. To avoid misunderstandings, and to accommodate situations where the concept of dimension is not available, we shall call this super space the *embedding space* below. Model sets have been found useful in numerous studies both by experimentalists modelling quasicrystals and by mathematicians studying aperiodic order and diffraction. One principal difficulty has been to find good characterizations of them. In particular, what are the intrinsic properties of a point set that permits its description as a projection from (parts of) some higher dimensional lattice?

Another major ingredient in the study of aperiodic point sets (and tilings) has been the use of dynamical systems. Given a (suitably discrete) point set $\varLambda \subset \mathbb{R}^d$, for example, one associates with it a space which is the closure of its $\mathbb{R}^d$-translation orbit, the closure taken in a topology that compares point sets for more or less exact match in local regions around the origin. This is called the dynamical hull, or *local hull* in this paper (since we shall meet other hulls that are dynamical systems as well). The major objective of this paper is to characterize model sets in terms of the properties of their local hulls.

As the theory of model sets and related mathematics has developed, it has become clear that the properties of the ambient space that are required are sufficiently weak that the group $\mathbb{R}^d$ may be replaced by any σ-compact locally compact Abelian (LCA) group G, without increasing the complexity of the proofs. In fact, this additional generality is to some extent necessary to understand model sets, as we shall see. In this paper, we take this more general setting.

The main theorem of the paper is:

Theorem 1. *Let G be a σ-compact LCA group and $(\mathbb{X}, G)$ a point set dynamical system on G. Then, for $(\mathbb{X}, G)$ to be the dynamical system associated to a repetitive regular model set it is necessary and sufficient for the following four conditions to be satisfied.*

1. *All elements of $\mathbb{X}$ are Meyer sets;*
2. $(\mathbb{X}, G)$ *is strictly ergodic, i.e., uniquely ergodic and minimal;*
3. $(\mathbb{X}, G)$ *has pure point dynamical spectrum with continuous eigenfunctions;*
4. *The eigenfunctions of* $(\mathbb{X}, G)$ *separate almost all points of* $\mathbb{X}$ (*i.e., the set* $\{\varLambda \in \mathbb{X} : \text{there exists } \varGamma' \neq \varGamma \text{ with } f(\varGamma) = f(\varGamma') \text{ for all eigenfunctions } f\}$ *has measure* 0).

The necessity of the conditions is already known [35, 36], so our main task is to deal with the converse – the four listed properties characterize repetitive regular model sets –, although we end up proving the necessity again in the process.

The proof is broken into three main parts. The first part is to use the properties (3) and (4) to identify elements of $\mathbb{X}$ that cannot be separated by the continuous eigenfunctions. This results in a new dynamical system $(\mathbb{E}, G)$, where $\mathbb{E}$ is actually a compact Abelian group, and a surjective G-mapping of $\mathbb{X}$ onto $\mathbb{E}$. Although this new group need not be a torus, it is nonetheless useful to simply call such a map a torus map or *torus parametrization*, in analogy to [1].

The second part is to show that $\mathbb{E}$ can be identified with another dynamical hull $\mathbb{A}$ of $\varLambda$, this time determined not by the local topology, but rather by a topology called the *autocorrelation topology*. This topology compares point sets globally for statistical match.

The third step, which actually appears first in the paper, is to show that a torus mapping of $\mathbb{X}$ onto $\mathbb{A}$ assures that we are in the situation of model sets – we can explicitly construct the embedding space, the lattice, and the mechanism which controls the projection down into the ambient space. This is really the heart of the matter. Given a Meyer set $\varLambda$, we have its two hulls $\mathbb{X}(\varLambda)$ and $\mathbb{A}(\varLambda)$. These are quite natural objects. The mapping β between them, when it exists, is the most natural one possible. It is really nothing but looking at the same elements of $\mathbb{X}(\varLambda)$, but in another topology. The assumption of the existence of the map is the same as saying that this change of topology is continuous, which in turn is the same as requiring that the local and global topologies are consistent with each other. It is this consistency that effectively characterizes the cut and project formalism.

The existence of windows for realizing the elements of $\mathbb{X}$ as model sets (or inter model sets) emerges as we require more out of the mapping β: first that it is one-to-one somewhere, and finally that it is one-to-one almost everywhere. If we go so far as to assume that it is one-to-one everywhere, we collapse into the crystallographic case (Theorem 10). Thus condition (4)

of Theorem 1 seems to contain the essence of aperiodicity (at least in the context of Meyer sets). This gives another instance for the intuition that regular model sets are a very natural generalization of crystallographic (i.e., fully periodic) point sets, and that aperiodic model sets are, in this sense, as close to periodic sets as possible among (repetitive) aperiodic Meyer sets.

Section 2 introduces the basic definitions and concepts used throughout the paper. In particular, in Paragraphs 2.1, 2.2 and 2.3, we establish the basic notions about the point sets and dynamical hulls that appear in the paper. Paragraph 2.4 deals with cut and project schemes and model sets. Paragraph 2.5 introduces the notion of a torus parametrization. While the material of that section is essentially known, the point of view taken there is of fundamental importance for our considerations.

Beyond the main theorem, there are a number of intermediate results that are interesting in their own right and are part of the overall proof. Section 3 serves the purpose of detailing these results and indicating the logical flow of the paper. The paper proper then begins with the consequences of a torus parametrization $\beta \colon \mathbb{X}(\varLambda) \to \mathbb{A}(\varLambda)$, gradually refining what can be learned from it as further conditions are added.

Model sets, as one sees them in the literature, come with varying definitions and side conditions, depending on the requirements of the moment. However, our results require quite precise notions of what constitutes a cut and project scheme, which windows are permitted, and how they relate to the cut and project scheme. Much, but not all, of this appears in the work of Schlottmann cited above. To make things clear, particularly the important ideas of irredundancy, which is not standard, and inter model sets, which are new [19], we have reworked this material and included it in the paper. Our attitude is that the main purpose of the paper is to prove the sufficiency direction of the Theorem 1, whence we have written the paper so that it moves in that direction from the very beginning. By the time that we have proved sufficiency, we actually know enough to prove necessity rather easily.

The paper has been delayed in reaching its final form by various circumstances around the lives of its authors. Nonetheless, its results have been announced in several places [26, 27]. Meanwhile, based on this paper, an extension of part of this theory to multi-colour sets has been worked out [19], and this has been effectively used in establishing the equivalence of pure pointedness and model sets for substitution tilings and point sets [18], a result that, for the case of unimodular Pisot substitutions in one dimension, has recently also been discussed in a slightly different context [16].

2. Basic definitions and hulls

This paper is a study of the relationship between various concepts in the regime of aperiodic order, formulated in terms of point sets in locally compact Abelian (LCA) groups. Let us first introduce the concepts.

2.1. **Aperiodic order and diffraction theory: the general setting.** Let G be a locally compact Abelian group, with Haar measure θ_G (normalized as $\theta_G(G) = 1$ if G is compact). We assume that G is σ-compact (also called countable at infinity). This is equivalent to the existence of an *averaging sequence* $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ of open, relatively compact sets $A_n \subset G$

with $\overline{A_n} \subset A_{n+1}$ and $G = \bigcup_{n\geq 1} A_n$. In fact, the averaging sequence can be chosen to be a *van Hove sequence*, see [36] for details. This means that, for every compact set $K \subset G$,

$$\lim_{n\to\infty} \frac{\theta_G\big(((K + A_n) \setminus A_n^{\circ}) \cup ((-K + \overline{G \setminus A_n}) \cap A_n)\big)}{\theta_G(A_n)} \;=\; 0,$$

where the bar (circle) denotes the closure (interior) of a set. In effect, this rather technical looking condition states that for each compact subset K of G, the K-boundary of the averaging sequence becomes negligible (in the sense of measure) to the sequence itself as $n \to \infty$. Note that general van Hove sequences need not be nested.

A subset $\varLambda$ of G is called *U-uniformly discrete* if, for the open neighbourhood U of 0 in G and for all $x \in \varLambda$, $(x+U) \cap \varLambda = \{x\}$. We say that $\varLambda$ is *uniformly discrete* if a neighbourhood U exists for which $\varLambda$ is U-uniformly discrete. By σ-compactness of G, every uniformly discrete set in G is countable. The set of all uniformly discrete subsets of G is denoted by $\mathcal{D} = \mathcal{D}(G)$ and the set of U-uniformly discrete subsets by $\mathcal{D}_U$.

Uniformly discrete sets can have various further regularity properties. A uniformly discrete subset $\varLambda$ of G is called *Delone* if it is also relatively dense, i.e., if there exists a compact set K in G with $G = \varLambda + K$.

Now, let $\varLambda$ be an arbitrary uniformly discrete set. Then, $\varLambda$ is of *finite local complexity* (FLC) if the set of K-clusters,

$$\{(-x + \varLambda) \cap K : x \in \varLambda\},$$

is finite for every compact $K \subset G$. This is equivalent to $\varLambda - \varLambda$ being discrete and closed [36]. If $\varLambda \subset G$ is a Delone set and there exists a finite set $F \subset G$ with $\varLambda - \varLambda \subset \varLambda + F$, then $\varLambda$ is called a *Meyer set*. Evidently, $\varLambda - \varLambda$ is uniformly discrete whenever $\varLambda$ is Meyer.

A point set $\varLambda \subset G$ of finite local complexity is called *repetitive* if for every compact K in G the set of repetitions of $\varLambda \cap K$

$$\{t \in G : (-t + \varLambda) \cap K = \varLambda \cap K\}$$

is relatively dense in G. $\varLambda$ is said to have *uniform patch frequencies* (some people say *uniform cluster frequencies*) if, for each finite subset P of G and for all $a \in G$,

$$\frac{\mathrm{card}\{t \in G : t + P \subset \varLambda \cap (a + B_n)\}}{\theta_G(B_n)}$$

converges uniformly in a along *every* van Hove sequence $\{B_n : n \in \mathbb{N}\}$, see [36] for details.

The diffraction pattern of a solid modelled by $\varLambda$ can be described as follows [10, 14]. For $x \in G$, let δ_x denote the normalized point (or Dirac) measure at $x \in G$. If the limit (taken in the vague topology)

$$\gamma \;:=\; \lim_{n\to\infty} \frac{1}{\theta_G(A_n)} \sum_{x,y \in \varLambda \cap A_n} \delta_{x-y}$$

exists, it is called the *autocorrelation measure* of $\varLambda$ relative to the averaging sequence $\mathcal{A}$. If $\varLambda$ has uniform patch frequencies, the limit exists and does not depend on the choice of $\mathcal{A}$ (as long as it is van Hove). The autocorrelation measure is positive definite and hence transformable, i.e., we can take its Fourier transform $\widehat{\gamma}$. This is a positive measure on the

dual group $\widehat{G}$, called the *diffraction measure*. For $G = \mathbb{R}^n$, it describes the outcome of a diffraction experiment, compare [10] for details.

2.2. **The local hull.** In this and the next paragraph, we introduce two topologies on the set $\mathcal{D}$ of all uniformly discrete subsets of G. The interplay of these two topologies is a main feature of the paper.

The so-called *local topology* (LT) on $\mathcal{D}$ is defined via the uniform structure given by the entourages

$$U_{\mathrm{LT}}(K,V) \ := \ \{(\Gamma,\Gamma') \in \mathcal{D}\times\mathcal{D} : (v+\Gamma)\cap K = \Gamma'\cap K \text{ for some } v \in V\}$$

for $K \subset G$ compact and V a neighbourhood of 0 in G. Thus, two uniformly discrete sets are close if they agree on a "large" compact set up to a "small" (global) translation. For definitions, terminology and basic theorems on uniformities, see [9, 31].

As is immediate from the definition of the local topology, the canonical action of G on $\mathcal{D}$ given by

$$G \times \mathcal{D} \longrightarrow \mathcal{D}, \quad (t, \Lambda) \mapsto -t + \Lambda,$$

is continuous. In particular, if $\mathbb{X} \subset \mathcal{D}$ is compact in the local topology and invariant under this action, then $(\mathbb{X}, G)$ is a topological dynamical system. Such a dynamical system will be called a *point set dynamical system*.

The hull of an element $\Lambda \in \mathcal{D}$ in the local topology (i.e., the closure of the orbit $G+\Lambda = \{x + \Lambda : x \in G\}$) is denoted by $\mathbb{X}(\Lambda)$.

Fact 1. [36] *If Λ is a Delone set, the hull $\mathbb{X}(\Lambda)$ is LT-compact if and only if Λ is of finite local complexity, i.e., if and only if $\Lambda - \Lambda$ is discrete and closed.* □

In this case, $\mathbb{X}(\Lambda)$ gives rise to a point set dynamical system $(\mathbb{X}(\Lambda), G)$. This dynamical system is a basic object in the study of the long-range order of discrete point sets because of its ability to reflect important geometric properties in the language of dynamical systems.

Fact 2. [36] *Let Λ be a Delone set of finite local complexity. Then, the dynamical system $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic (i.e., there exists precisely one G-invariant probability measure on $\mathbb{X}(\Lambda)$) if and only if Λ has uniform patch frequencies.* □

Two Delone sets Λ, Λ' are *locally indistinguishable* (LI) if each cluster of Λ (i.e., each set of the form $\Lambda \cap K$ with $K \subset G$ compact) is a translate $-x + (\Lambda' \cap (x + K))$ of a cluster of Λ' and vice versa. This equivalence relation defines the so-called LI classes, and one has

Fact 3. *Let G be an LCA group and $\Lambda \subset G$ a Delone set of finite local complexity. Then, the following properties are equivalent.*

(1) *The set Λ is repetitive.*
(2) *The hull $\mathbb{X}(\Lambda)$ is the LI class of Λ.*
(3) *The dynamical system $(\mathbb{X}(\Lambda), G)$ is minimal.*

Proof. This is a variant of Gottschalk's theorem, see [36] for details. □

The definition of closeness in the local topology has a special consequence for translates of Meyer sets.

Fact 4. *Let $\varLambda$ be a Meyer set. Then, for all suitably small neighbourhoods V of 0 in G and all compact $C \subset G$ with $C \cap \varLambda \neq \varnothing$, the equality $\varLambda \cap C = (-x+\varLambda) \cap C$ holds whenever $(-x+\varLambda, \varLambda) \in U_{\mathrm{LT}}(C, V)$ for $x \in \varLambda - \varLambda$.*

Proof. As $\varLambda$ is Meyer, one has $\varLambda - \varLambda \subset \varLambda + F$ with F a finite set. Clearly, also $(\varLambda-\varLambda)+(\varLambda-\varLambda) \subset \varLambda + F'$, with F' still finite, so that uniform discreteness persists to $\varLambda - \varLambda + (\varLambda - \varLambda)$. Thus, there exists an open neighbourhood V of 0 in G so small that $V \cap ((\varLambda-\varLambda)+(\varLambda-\varLambda)) = \{0\}$. Now, $(-x+\varLambda, \varLambda) \in U_{\mathrm{LT}}(C, V)$ for $x \in G$ implies

$$(v - x + \varLambda) \cap C = \varLambda \cap C$$

for some $v \in V$. Now, if $x \in \varLambda - \varLambda$, then $\varLambda \cap C \neq \varnothing$ yields $v \in V \cap ((\varLambda-\varLambda)+(\varLambda-\varLambda)) = \{0\}$ and the fact is proved. $\square$

Let $(\mathbb{X}, G)$ be a point set dynamical system which is uniquely ergodic. In this case, there is a canonical Hilbert space associated to $(\mathbb{X}, G)$, the space $L^2(\mathbb{X}, \mu)$ of square integrable functions on $\mathbb{X}$ (with respect to the unique G-invariant probability measure μ). The action of G on $\mathbb{X}$ gives rise to a unitary representation T of G on this space via

$$T_t \colon\; L^2(\mathbb{X}, \mu) \;\longrightarrow\; L^2(\mathbb{X}, \mu)\,, \quad (T_t f)(\varLambda) \;:=\; f(-t+\varLambda)\,,$$

for $f \in L^2(\mathbb{X}, \mu)$ and $t \in G$. An $f \in L^2(\mathbb{X}, \mu)$ is called an *eigenfunction* of T with *eigenvalue* $\hat{s} \in \widehat{G}$ (the dual group) if $T_t f = (\hat{s}, t) f$ for every $t \in G$, where $(\hat{s}, .)$ denotes the character defined by $\hat{s}$. An eigenfunction (to $\hat{s}$, say) is called *continuous* if it has a continuous representative f with $f(-t+\varLambda) = (\hat{s}, t) f(\varLambda)$, for all $\varLambda \in \mathbb{X}$ and $t \in G$. The representation T is said to have *pure point spectrum* if the set of eigenfunctions is total in $L^2(\mathbb{X}, \mu)$. One then also says that the dynamical system $(\mathbb{X}, G)$ has *pure point dynamical spectrum*.

2.3. **The autocorrelation hull.** The *upper density* of a point set $\varLambda \subset G$ is defined by

$$\overline{\mathrm{dens}}(\varLambda) \;:=\; \limsup_{n\to\infty} \frac{\mathrm{card}(\varLambda \cap A_n)}{\theta_G(A_n)}$$

with respect to the averaging van Hove sequence $\mathcal{A}$ chosen before. The lower density, $\underline{\mathrm{dens}}(\varLambda)$, is defined analogously. If $\overline{\mathrm{dens}}(\varLambda) = \underline{\mathrm{dens}}(\varLambda)$, this is called the *density* of $\varLambda$, denoted by $\mathrm{dens}(\varLambda)$. We shall usually suppress the explicit reference to $\mathcal{A}$.

The *mixed autocorrelation topology* (mACT) on $\mathcal{D}$ is defined via the uniform structure given by the entourages

$$U_{\mathrm{mACT}}(V, \varepsilon) \;:=\; \{(\varGamma, \varGamma') \in \mathcal{D} \times \mathcal{D} : d(v+\varGamma, \varGamma') \leq \varepsilon \text{ for some } v \in V\},$$

for every neighbourhood V of 0 in G and every $\varepsilon > 0$, where the pseudo-metric d on $\mathcal{D}$ is defined by the upper density of the symmetric difference of sets:

$$d(\varGamma, \varGamma') \;:=\; \overline{\mathrm{dens}}\,(\varGamma \,\triangle\, \varGamma')\,. \tag{1}$$

Note that the triangle inequality follows from the fact that $\varGamma \triangle \varGamma' \subset (\varGamma \triangle \varGamma'') \cup (\varGamma' \triangle \varGamma'')$, for arbitrary point sets $\varGamma'' \subset \mathcal{D}$. With this definition, d is G-invariant (i.e., $d(t+\varGamma, t+\varGamma') = d(\varGamma, \varGamma')$ for all $t \in G$ and all $\varGamma, \varGamma' \in \mathcal{D}$), because $\mathcal{A}$ has the van Hove property. We call mACT the *mixed* autocorrelation topology because it mixes the ordinary topology of G with the topology introduced by the pseudo-metric d. The topology induced by d itself, in turn,

ultimately arises from the autocorrelation (see below) and we thus call it the *autocorrelation topology.*

Note that d contains information on statistical coincidence of the global structure. Thus, two sets are close in the mixed autocorrelation topology if their global structures are statistically close up to a small translation.

It is obvious that, for a general LCA group G, d does not define a metric on $\mathcal{D}$. However, it does permit the construction of a completion where d becomes a metric. To see this, fix an open neighbourhood U of 0 in G and consider the restriction of the pseudo-metric d (still denoted by d) to $\mathcal{D}_U$. Introduce an equivalence relation $\equiv$ on $\mathcal{D}_U$ by setting $\varGamma \equiv \varGamma'$ if and only if $d(\varGamma, \varGamma') = 0$, with d as defined in (1). The quotient of $\mathcal{D}_U$ by this equivalence relation is denoted by $\mathcal{D}_U^{\equiv}$. By construction, the pseudo-metric d on $\mathcal{D}$ induces a metric on $\mathcal{D}_U^{\equiv}$, which we again call d. Then, d is a G-invariant metric on $\mathcal{D}_U^{\equiv}$, and $\mathcal{D}_U^{\equiv}$ is complete as a metric space, though neither of these two facts is obvious, compare [28, Cor. 3.10].

We give $\mathcal{D}_U^{\equiv}$ the uniform topology induced by U_{mACT} and again call it the *mixed autocorrelation topology.* Again, this is *not* the same as the metric topology induced by d itself, since it takes small shifts in the sets into account in order to make the action of G continuous.

Proposition 1. *$\mathcal{D}_U^{\equiv}$ is complete in the mixed autocorrelation topology.*

Proof. For $\varLambda, \varLambda' \in \mathcal{D}_U$, if $\varepsilon > 0$ and $d(\varLambda, \varLambda') < \varepsilon$, one finds

$$\begin{aligned}
\varepsilon \;&>\; d(\varLambda, \varLambda') \;=\; \overline{\mathrm{dens}}\,(\varLambda \triangle \varLambda') \\
&=\; \overline{\mathrm{dens}}\,\big(\big(\varLambda \backslash (\varLambda \cap \varLambda')\big) \,\cup\, \big(\varLambda' \backslash (\varLambda \cap \varLambda')\big)\big) \\
&\geq\; \overline{\mathrm{dens}}\,\big(\varLambda \backslash (\varLambda \cap \varLambda')\big) \;\geq\; \overline{\mathrm{dens}}\,(\varLambda) - \underline{\mathrm{dens}}\,(\varLambda \cap \varLambda')\,.
\end{aligned}$$

By symmetry, one also has $\varepsilon > \overline{\mathrm{dens}}\,(\varLambda') - \underline{\mathrm{dens}}\,(\varLambda \cap \varLambda')$, and hence $|\,\overline{\mathrm{dens}}\,(\varLambda) - \overline{\mathrm{dens}}\,(\varLambda')| < 2\varepsilon$.

Now, let $\{\varLambda_i\} \subset \mathcal{D}_U$ be a Cauchy net with respect to the mixed autocorrelation topology. We want to prove that the net converges when seen in $\mathcal{D}_U^{\equiv}$.

For any $\varepsilon > 0$ and any open neighbourhood V of 0, there is some n so that for all $i, j \succcurlyeq n$ (with $\succcurlyeq$ referring to the partial order on the index set) and for suitable $v_{ij} \in V$, one has $d(v_{ij} + \varLambda_i, \varLambda_j) < \varepsilon$. Then, by the above calculation, $|\,\overline{\mathrm{dens}}\,(v_{ij} + \varLambda_i) - \overline{\mathrm{dens}}\,(\varLambda_j)| < 2\varepsilon$. Since $\overline{\mathrm{dens}}\,(v_{ij} + \varLambda_i) = \overline{\mathrm{dens}}\,(\varLambda_i)$, we see that $\{\,\overline{\mathrm{dens}}\,(\varLambda_i)\}$ is a Cauchy net in $\mathbb{R}$ and so converges to some limit $c \geq 0$. If $c = 0$, then $\{\varLambda_i\} \to \varnothing \in \mathcal{D}_U^{\equiv}$, and we are done. So we only need to consider the case that $c > 0$.

Returning to the Cauchy net $\{\varLambda_i\} \subset \mathcal{D}_U$, choose any open neighbourhood V of 0 so that $-V + V + V \subset U$, and any ε that satisfies $0 < \varepsilon < c/3$. Fix n so that $i, j \succcurlyeq n$ implies that $(\varLambda_i, \varLambda_j) \in U_{\mathrm{mACT}}(V, \varepsilon)$ and $\overline{\mathrm{dens}}\,(\varLambda_i) > c/2$.

We know that, for all $j, k \succcurlyeq n$,

$$d(v_{jk} + \varLambda_j, \varLambda_k) < \varepsilon\,, \quad \text{for some } v_{jk} \in V.$$

Then, for all $j, k \succcurlyeq n$, $d(v_{jk} + v_{nj} + \varLambda_n, v_{nk} + \varLambda_n) < 3\varepsilon$, or, using translation invariance, $d(-v_{nk} + v_{jk} + v_{nj} + \varLambda_n, \varLambda_n) < 3\varepsilon$. However, for $x \in \varLambda_n$,

$$\{-v_{nk} + v_{jk} + v_{nj} + x\} \cap \varLambda_n = \begin{cases} \{x\}, & \text{if } -v_{nk} + v_{jk} + v_{nj} = 0, \\ \varnothing, & \text{otherwise,} \end{cases}$$

because $-V+V+V \subset U$ and all the sets $\varLambda_\ell$ lie in $\mathcal{D}_U$. So, if $-v_{nk}+v_{jk}+v_{nj} \neq 0$, then $d(-v_{nk}+v_{jk}+v_{nj}+\varLambda_n, \varLambda_n) = 2\,\overline{\mathrm{dens}}\,(\varLambda_n) \geq 2(c/2) > 3\varepsilon$, a contradiction. Thus $v_{jk}+v_{nj} = v_{nk}$ and $v_{jk} = -v_{kj}$ for all j,k.

In principle, v_{jk} depends on V and ε. However, the same little argument shows that it is actually unique in the sense that it will be the same element for any $V' \subset V$, $\varepsilon \leq \varepsilon'$.

Let $\varLambda'_j := -v_{nj} + \varLambda_j \in \mathcal{D}_U$, for all $j \succcurlyeq n$. Then, for all $j,k \succcurlyeq n$,

$$d(\varLambda'_j, \varLambda'_k) = d(-v_{nj}+\varLambda_j, -v_{nk}+\varLambda_k) = d(v_{jk}+\varLambda_j, \varLambda_k) < \varepsilon\,.$$

This shows that $\{\varLambda'_n\}$ is a Cauchy net in $\mathcal{D}_U$, with respect to the metric topology defined by d. By [28, Cor. 3.9], it converges to some $\varLambda \in \mathcal{D}_U^{\equiv}$. It is easy to see that also $\{\varLambda_n\}$ converges to $\varLambda$, which completes the argument. $\square$

Denote the equivalence class of $\varLambda \in \mathcal{D}_U$ by $[\varLambda]$, and let β be the canonical mapping from $\mathcal{D}_U$ to $\mathcal{D}_U^{\equiv}$, i.e.,

$$\beta :\ \mathcal{D}_U \longrightarrow \mathcal{D}_U^{\equiv}\,, \quad \varLambda \mapsto [\varLambda]\,.$$

Each $\varLambda$ in $\mathcal{D}_U$ gives rise to the *autocorrelation hull* $\mathbb{A}(\varLambda)$ defined as the closure of the orbit $G+\beta(\varLambda)$ in the mixed autocorrelation topology. By construction, one may as well consider $\mathbb{A}(\varLambda)$ to be the Hausdorff completion of G with respect to the uniform topology on G that is given by pulling back the autocorrelation topology from $\mathcal{D}_U$. In detail, define a pseudo-metric (relative to $\varLambda$) on G by

$$d_G(s,t) \;:=\; d(t+\varLambda, s+\varLambda) \;=\; d(t-s+\varLambda, \varLambda). \tag{2}$$

Then, the uniformity on G is described by the sets

$$\{(t,s) \in G \times G \,:\, d(v+t+\varLambda, s+\varLambda) < \varepsilon\} \tag{3}$$

where $v \in V$, and V and ε run over all neighbourhoods of 0 and all non-negative real numbers, respectively.

This can be written in a more suggestive way via the set of ε-*almost periods* of $\varLambda$,

$$P_\varepsilon \;:=\; \{t \in \varLambda - \varLambda : d_G(t,0) < \varepsilon\}\,. \tag{4}$$

Then, the entourages (3) are just the sets

$$\{(t,s) \in G \times G \,:\, t-s \in V + P_\varepsilon\}\,. \tag{5}$$

These entourages are evidently G-invariant. This has an important consequence: $\mathbb{A}(\varLambda)$, now being the completion of the Abelian group G with respect to the invariant uniformity as defined by (5), carries a natural Abelian group structure. Moreover, G acts minimally on $\mathbb{A}(\varLambda)$ through the translation action. This is the second topological dynamical system for our group G. Of course, this construction depends entirely (and crucially) on the starting set $\varLambda$. Below, we shall often shift back and forth between the two views of $\mathbb{A}(\varLambda)$: as a subset of $\mathcal{D}_U^{\equiv}$ and as a completion of G.

If we start with a set $\varLambda \in \mathcal{D}_U$, we can form the two hulls $\beta(\mathbb{X}(\varLambda))$ and $\mathbb{A}(\varLambda)$. In general, these are *not* related in any obvious way. In particular, neither is contained in the other. If, however, β is continuous, then obviously $\beta(\mathbb{X}(\varLambda)) \subset \mathbb{A}(\varLambda)$. We refer to this mapping $\beta :\ \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$ as the *canonical torus map*. Moreover, if β is continuous and $\mathbb{X}(\varLambda)$ is compact, then $\beta(\mathbb{X}(\varLambda)) = \mathbb{A}(\varLambda)$, as $\beta(\mathbb{X}(\varLambda))$ is then a compact, and hence closed, set

containing $G + \beta(\Lambda)$. In this case, $\mathbb{A}(\Lambda)$ becomes a compact topological group. We shall have more to say about this situation.

2.4. **Cut and project schemes and model sets.** Here, we introduce model sets and discuss some of their basic features. For further details and proofs, we refer to [24, 35, 36].

Model sets arise as (partial) projections from a high dimensional periodic structure to a lower dimensional subspace. This is formalized in the following notion.

A *cut and project scheme*, or CPS for short, is a triple $(G, H, \mathcal{L})$ consisting of locally compact Abelian (LCA) groups G and H, with G also being σ-compact, and a lattice $\mathcal{L}$ in $G \times H$ such that the two natural projections $\pi_1 \colon G \times H \longrightarrow G$, $(t, h) \mapsto t$ and $\pi_2 \colon G \times H \longrightarrow H$, $(t, h) \mapsto h$ of the scheme

$$\begin{array}{ccccc} G & \xleftarrow{\pi_1} & G \times H & \xrightarrow{\pi_2} & H \\ & & \cup & & \\ & & \mathcal{L} & & \end{array} \tag{6}$$

satisfy the following properties:

- The restriction $\pi_1|_{\mathcal{L}}$ of π_1 to $\mathcal{L}$ is injective.
- The image $\pi_2(\mathcal{L})$ is dense in H.

Let $L := \pi_1(\mathcal{L})$ and $(.)^\star \colon L \longrightarrow H$ be the mapping $\pi_2 \circ (\pi_1|_{\mathcal{L}})^{-1}$. Note that * is indeed well defined on L and that it can often be extended to a larger subgroup of G (such as the rational span $\mathbb{Q}L$ in the Euclidean case), but not to all of G.

Moreover, as $\mathcal{L}$ is a discrete and co-compact subgroup of $G \times H$, the quotient

$$\mathbb{T} \; := \; (G \times H)/\mathcal{L}$$

is a compact Abelian group. In the standard cut and project setting with Euclidean spaces only, this group is a torus, compare [1]. There is an obvious action of G on $\mathbb{T}$ given by

$$x + \big((t, h) + \mathcal{L}\big) \; := \; (x + t, h) + \mathcal{L}\,, \quad x \in G\,.$$

Then, $(\mathbb{T}, G)$ is minimal and hence uniquely ergodic as well (as $\mathbb{T}$ is a compact Abelian group).

Given a CPS (6) and a subset $S \subset H$, we define $\curlywedge(S)$ by

$$\curlywedge(S) \; := \; \{x \in L : x^\star \in S\}.$$

Then, $\curlywedge(S)$ is relatively dense if the interior of S is non-empty and it is uniformly discrete if the closure of S is compact, see [24] for details.

A *model set*, associated with the CPS (6), is a non-empty subset Λ of G of the form

$$\Lambda \; = \; x + \curlywedge(y + W),$$

where $x \in G$, $y \in H$, and $W \subset H$ is a non-empty compact set with $W = \overline{W^\circ}$. A model set $\Lambda = x + \curlywedge(y + W)$ is called *regular* if $\theta_H(\partial W) = 0$. A (regular) model set of the above form is called *generic* if $(y + \partial W) \cap L^\star = \varnothing$. Any model set is a Delone set. Namely, it is uniformly discrete as W is compact and relatively dense as W has nonempty interior. In fact, they are even Meyer sets, because $\Lambda - \Lambda \subset \curlywedge(W - W)$ and $W - W$ is compact, too, and they are thus also FLC sets. Moreover, a regular model set has uniform patch frequencies (i.e., the associated dynamical system is uniquely ergodic) and a generic model set is repetitive.

Our prime concern are model sets and their dynamical systems. It turns out that the dynamical system associated with the model set $\curlywedge(W)$ may contain sets $\varLambda'$ which are not model sets themselves with respect to the given CPS. It is hard to determine their precise structure in terms of the window. However, under a condition called irredundancy (see below for more), all of these sets $\varLambda'$ satisfy

$$t + \curlywedge(c + W^{\circ}) \; \subset \; \varLambda' \; \subset \; t + \curlywedge(c + W) \tag{7}$$

with suitable $t \in G$ and $c \in H$. This suggests to work right from the start with sets of the form $t + \curlywedge(W^{\circ}) \subset \varLambda \subset t + \curlywedge(W)$. This approach is also taken in [19] in order to characterize multi-component model sets. We shall call such sets *inter model sets*, or IMS for short.

The condition we need reads as follows (see [19] and Sections 5 and 9).

Definition 1. Let $(G, H, \mathcal{L})$ be a CPS. A subset S of H is called *irredundant* (with respect to the given CPS), if its stabilizer in H is trivial, i.e., if the equation $c + S = S$ holds only for $c = 0 \in H$.

To state our results, we also need the following definition.

Definition 2. A dynamical system $(\mathbb{X}, G)$ is said to be associated with a (regular) model set if there exists a (regular) model set $\varLambda$ such that $\mathbb{X} = \mathbb{X}(\varLambda)$.

2.5. **The torus parametrization: Abstract results.** In this paragraph, we look briefly at factors of dynamical systems $(\mathbb{X}, G)$ in which the factors are of the form of compact Abelian groups with minimal G-actions. These results are essentially known. Throughout, G will be an LCA group (although most of this works for other groups as well). The situations that we have in mind are special actions of G by translations on point set dynamical systems. These actions generalize concepts from [1] and [36] known as torus parametrizations, and we retain this terminology here.

Definition 3. Let $\mathbb{X}$ be a compact space and $(\mathbb{X}, G)$ a topological dynamical system under the action of G. A continuous G-map $\rho\colon \mathbb{X} \longrightarrow \mathbb{K}$ into a compact Abelian group $\mathbb{K}$ on which G acts minimally is called *torus parametrization*.

Definition 4. Let $\rho\colon \mathbb{X} \longrightarrow \mathbb{K}$ be a torus parametrization. For $\xi \in \mathbb{K}$, we call the inverse image $\rho^{-1}(\{\xi\})$ the *fibre* over ξ. Then, $\varGamma \in \mathbb{X}$ is called *singular* if the fibre over $\rho(\varGamma)$ consists of more than one element. Otherwise, $\varGamma$ is called *non-singular*. In this case, $\{\varGamma\} = \rho^{-1}(\rho(\varGamma))$ is called a *singleton fibre*.

Lemma 1. *If $\rho : \mathbb{X} \longrightarrow \mathbb{K}$ is a torus parametrization, ρ is onto.*

Proof. As $\mathbb{X}$ is compact and ρ continuous, the image $\rho(\mathbb{X})$ is compact. Let $\varGamma$ be an arbitrary element of $\mathbb{X}$. As ρ is a G-map, $\rho(\mathbb{X})$ contains the orbit of $\rho(\varGamma)$. As G acts minimally on $\mathbb{K}$, this orbit is dense in $\mathbb{K}$. Thus, $\rho(\mathbb{X})$ is a dense compact subset of $\mathbb{K}$, hence agrees with $\mathbb{K}$. □

Let us continue with an interesting property of the torus parametrization. Namely, each torus parametrization induces a minimal subsystem of the original dynamical system.

Proposition 2. *Let $\rho\colon \mathbb{X} \longrightarrow \mathbb{K}$ be a torus parametrization. If the set*

$$R(\mathbb{X}) \; := \; \{\varGamma \in \mathbb{X} : \varGamma \; \textit{ is non-singular}\}$$

is non-empty, it is G-invariant, and G acts minimally on its closure $\mathbb{X}_R := \overline{R(\mathbb{X})}$.

Proof. The G-invariance of $\mathbb{X}_R$ is clear, as ρ is a G-map; it remains to show minimality. To do so, let an arbitrary $\Gamma \in R(\mathbb{X})$ be given, and consider some $\Lambda' \in \mathbb{X}_R$. Let $\mathbb{X}(\Lambda')$ be the closure of its G-orbit in $\mathbb{X}$. Of course, the restriction $\rho_{\mathbb{X}(\Lambda')} \colon \mathbb{X}(\Lambda') \longrightarrow \mathbb{K}$ of ρ to $\mathbb{X}(\Lambda')$ is a torus parametrization as well. In particular, it is onto. Thus, we can find $\Gamma' \in \mathbb{X}(\Lambda')$ with $\rho(\Gamma') = \rho(\Gamma)$. By $\Gamma \in R(\mathbb{X})$, we infer $\Gamma = \Gamma' \in \mathbb{X}(\Lambda')$. As $\Gamma \in R(\mathbb{X})$ was arbitrary, this implies $R(\mathbb{X}) \subset \mathbb{X}(\Lambda')$. As $\mathbb{X}(\Lambda') \subset \mathbb{X}_R$ is clear anyway, we obtain, after taking closures,

$$\mathbb{X}_R \,\subset\, \mathbb{X}(\Lambda') \,\subset\, \mathbb{X}_R \, .$$

As $\Lambda' \in \mathbb{X}_R$ was arbitrary, the statement follows. □

We now discuss continuity properties of the inverse of a torus parametrization. While these results are not particularly hard to prove, they are a crucial ingredient behind the reconstruction of the window given in Lemma 3 in Section 5.

Proposition 3. *Let $\rho \colon \mathbb{X} \longrightarrow \mathbb{K}$ be a torus parametrization. Let $\alpha \colon \mathbb{K} \longrightarrow \mathbb{X}$ be any section of ρ (i.e., $\rho \circ \alpha$ is the identity on $\mathbb{K}$). Then, α is continuous at all points which are images of non-singular points, i.e., at all points of $\rho(R(\mathbb{X}))$.*

The proof of this proposition is an immediate consequence of the following lemma.

Lemma 2. *Let K_1 and K_2 be compact spaces and $\sigma \colon K_1 \longrightarrow K_2$ continuous. Let $\xi_1 \in K_1$ and $\xi_2 \in K_2$ be given such that $\{\xi_1\} = \sigma^{-1}(\{\xi_2\})$. Then, a net (ξ_ι) in K_1 converges to ξ_1 whenever $(\sigma(\xi_\iota))$ converges to ξ_2.*

Proof. By compactness of K_1, the net (ξ_ι) has converging subnets. Thus, it suffices to show that every converging subnet converges to ξ_1. So, consider a converging subnet. Without loss of generality, we may assume this converging subnet to be (ξ_ι) itself. Let ξ_1' be its limit. Then, by continuity of σ, we have $\sigma(\xi_1') = \lim_\iota \sigma(\xi_\iota) = \xi_2$. As, by assumption, $\{\xi_1\} = \sigma^{-1}(\{\xi_2\})$, we infer $\xi_1' = \xi_1$, and the proof (both of Lemma 2 and Proposition 3) is complete. □

3. Outline of the paper and summary of the main theorems

The overall objective of the paper is to prove Theorem 1, particularly in the direction of sufficiency. The basic setting is that of a Meyer set Λ for which the local hull $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic. We are interested in continuous G-mappings from the local hull to the autocorrelation hull, and particularly in those that are non-singular almost everywhere. Simply the existence of such a mapping $\beta_{\mathbb{A}} \colon \mathbb{X}(\Lambda) \longrightarrow \mathbb{A}(\Lambda)$ produces the first prerequisite for the appearance of model sets, a cut and project scheme. This is described in Section 4.2. Any cut and project scheme (CPS) has associated with it a compact Abelian group $\mathbb{T}$ – the quotient of the product of the ambient group and the internal group by the associated lattice. A key feature of the cut and project scheme that arises in our situation is that the mapping $\beta_{\mathbb{A}} \colon \mathbb{X}(\Lambda) \longrightarrow \mathbb{A}(\Lambda)$ can be viewed as a mapping $\mathbb{X}(\Lambda) \longrightarrow \mathbb{T}$:

Theorem 2. *Let Λ be a Meyer set for which $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic. Suppose that there exists a continuous G-map $\beta_{\mathbb{A}} \colon \mathbb{X}(\Lambda) \longrightarrow \mathbb{A}(\Lambda)$. Then, there is a CPS $(G, H, \mathcal{L})$ with associated compact Abelian group $\mathbb{T}$ for which $\mathbb{A} \simeq \mathbb{T}$ via a topological isomorphism which is a G-map that sends $\Lambda \in \mathbb{A}$ to $0 \in \mathbb{T}$. In particular, there is a G-map $\beta_{\mathbb{T}} \colon \mathbb{X}(\Lambda) \longrightarrow \mathbb{T}$.*

Having constructed a cut and project set, we next need a window to be in the regime of model sets. As studied in Section 5, the crucial condition to provide a window is non-singularity of the G-map $\beta_{\mathbb{A}}$. To avoid technical difficulties, we state the result here in a slightly simplified form.

Theorem 3A. *Let $\varLambda$ be a Meyer subset of G such that $(\mathbb{X}(\varLambda), G)$ is uniquely ergodic. Assume that there exists a continuous G-map $\beta_{\mathbb{A}} : \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$ which is one-to-one at least at one point. Then, there is a minimal dynamical subsystem $(\mathbb{X}(\varLambda)_R, G)$ of $(\mathbb{X}(\varLambda), G)$ that is associated with a repetitive model set. In particular, if $\varLambda$ is repetitive, $(\mathbb{X}(\varLambda), G)$ itself is associated with a model set.*

The previous theorem does not assert that the constructed model set is regular, i.e., that the measure of the boundary of the window is 0. Concerning this topic, our result is Theorem 5. It shows that the boundary has Haar measure 0 if and only if the map $\beta_{\mathbb{A}}$ is one-to-one almost everywhere. In fact, if the canonical map $\beta\colon \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$ is one-to-one almost everywhere, we can get further:

Theorem 6. *Let G be a σ-compact LCA group and $\varLambda$ a Meyer subset of G such that the canonical map $\beta\colon \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$ is continuous and one-to-one almost everywhere, with respect to the Haar measure on $\mathbb{A}(\varLambda) = \beta(\mathbb{X}(\varLambda))$. Then, $\mathbb{X}(\varLambda)$ is uniquely ergodic and $\varLambda$ agrees with a regular model set up to a set of density* 0. *Furthermore, if $\varLambda$ is repetitive, $\mathbb{X}(\varLambda)$ is actually associated to a regular model set.*

So far, we have assumed existence of a continuous G-map $\beta_{\mathbb{A}} : \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$. But what conditions are required to obtain such a map? This is studied in Section 6. Our main answer is the following.

Theorem 7. *Let $\varLambda$ be a Meyer subset of G such that $(\mathbb{X}(\varLambda), G)$ is uniquely ergodic. Then, the following assertions are equivalent.*

(a) *There exists a continuous G-map $\beta_{\mathbb{A}} : \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$.*
(b) *$(\mathbb{X}(\varLambda), G)$ has pure point dynamical spectrum with continuous eigenfunctions.*

In this case, $\varGamma, \varGamma' \in \mathbb{X}(\varLambda)$ satisfy $\beta_{\mathbb{A}}(\varGamma) = \beta_{\mathbb{A}}(\varGamma')$ if and only if $f(\varGamma) = f(\varGamma')$ for every eigenfunction f.

The proof of the implication (b) $\Longrightarrow$ (a) of this theorem requires an intermediate step. From the assumptions on $(\mathbb{X}(\varLambda), G)$, we create a new dynamical system $(\mathbb{E}, G)$ by identifying elements of $\mathbb{X}$ which are indistinguishable by means of the continuous eigenfunctions. This new space $\mathbb{E}$ can be given the structure of a compact Abelian group. This new group is then shown to be just $\mathbb{A}(\varLambda)$. This is discussed in Section 7 and, in particular, in Theorem 8.

Theorems 3, 5 and 7 establish the sufficiency part of our main Theorem 1 (and most of the necessity too). This is discussed in Section 8. The link back is provided in Section 9 via the following result.

Theorem 9. *Let a CPS $(G, H, \mathcal{L})$ and a non-empty window $W \subset H$ with $W = \overline{W^{\circ}}$ and $\theta_H(\partial W) = 0$ be given. If $\varLambda \subset G$ satisfies $t + \curlywedge(W^{\circ}) \subset \varLambda \subset t + \curlywedge(W)$ for some $t \in G$, then the canonical map $\beta : \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$ is continuous and one-to-one almost everywhere.*

Let us make a short comment here: During the process of proving the above results, we encounter groups $\mathbb{A}$ and $\mathbb{T}$ and maps $\beta_{\mathbb{A}}$ and $\beta_{\mathbb{T}}$ from $\mathbb{X}(\varLambda)$ into these groups. We show that

these groups are isomorphic and that, in this sense, $\beta_{\mathbb{A}}$ and $\beta_{\mathbb{T}}$ agree. In fact, in retrospect, we can then even show that these maps agree with the canonical map β introduced above. However, this is not at all clear at the respective times of appearance and, for this reason, we carefully distinguish these maps and groups.

Finally, our results also imply an interesting characterization of the fully periodic case as discussed in Section 10:

Definition 5. A set $\varLambda \subset G$ is called *crystallographic* (or fully periodic) if its set of periods

$$\mathrm{per}(\varLambda) \ := \ \{t \in G : t + \varLambda = \varLambda\}$$

forms a *lattice*, i.e., a co-compact discrete subgroup of G.

Theorem 10. *Let G be an LCA group and $\varLambda$ a uniformly discrete subset of G. Then, the following assertions are equivalent.*

(i) *$\varLambda$ is crystallographic.*
(ii) *$\varLambda$ is Meyer and the map $\beta\colon\ \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$ is continuous and injective.*
(iii) *All of the following conditions hold:*
  (1) *All elements of $\mathbb{X}(\varLambda)$ are Meyer sets.*
  (2) *$(\mathbb{X}(\varLambda), G)$ is uniquely ergodic.*
  (3) *$(\mathbb{X}(\varLambda), G)$ has pure point dynamical spectrum with continuous eigenfunctions.*
  (4) *The eigenfunctions separate all points of $\mathbb{X}(\varLambda)$.*

In this case, $(\mathbb{X}(\varLambda), G)$ is also minimal, hence strictly ergodic.

The paper revolves around the important concept of Meyer sets. We have defined a set $\varLambda \subset G$ to be Meyer if it is a Delone set and $\varLambda - \varLambda$ is contained in a finite number of translates of $\varLambda$. We already noted that this implies that $\varLambda - \varLambda$ is also a Delone set (the important point being that it is uniformly discrete). For $G = \mathbb{R}^d$, this is an equivalence, and in fact the most common definition of a Meyer set is a Delone set whose set of differences is uniformly discrete. This result is due to Lagarias [17]. In the Appendix, we show that the two concepts are equivalent if G is compactly generated. We also show that, in this case, the requirement that $\varLambda - \varLambda$ be uniformly discrete is equivalent to the apparently weaker statement that for each compact subset K of G, the number of points of $(t+K) \cap (\varLambda - \varLambda)$ is finite and uniformly bounded as t runs over G (Theorem 11).

4. Consequences of a continuous G-mapping $\beta_{\mathbb{A}}\colon\ \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$: Construction of a cut and project scheme

Let $\varLambda$ be a Meyer subset of G such that the associated dynamical system $(\mathbb{X}(\varLambda), G)$ is uniquely ergodic. As $\varLambda$ is Meyer, there is an open neighbourhood U of 0 in G so that $\varLambda$ is U-uniformly discrete, i.e., $\varLambda \in \mathcal{D}_U$. Moreover, it also follows that each element of $\mathbb{X}(\varLambda)$ is U-uniformly discrete, too. As discussed in Section 2.3, $\varLambda$ gives rise to the autocorrelation hull $\mathbb{A}$, which is an Abelian group.

In this section, we assume that $\mathbb{A}(\varLambda)$ is compact and that there exists a torus parametrization $\beta_{\mathbb{A}} :\ \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$. We do *not* assume that the map $\beta_{\mathbb{A}}$ is given by the canonical projection β.

Our objective in this section is to create a cut and project scheme out of this torus mapping and to show that $\mathbb{A}(\varLambda)$ is G-isomorphic with the torus $\mathbb{T}$ of the associated cut and project scheme. Section 5 then shows how non-singularity of the torus parametrization can be used to provide and study a window.

Below, we shall freely use notation from Section 2 and, in particular, Paragraph 2.4.

4.1. **Establishing Axioms (A1) – (A4) of** [5]**.** Let $\varLambda$ be a Meyer subset of G such that the associated dynamical system $(\mathbb{X}(\varLambda), G)$ is uniquely ergodic. We assume the existence of a torus parametrization $\beta_{\mathbb{A}} \colon \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$.

In order to create a CPS from this data, we rely on the construction described in [5], based on the Dirac comb $\delta_\varLambda$ of our point set $\varLambda$. It is defined by $\delta_\varLambda := \sum_{x\in\varLambda} \delta_x$. The construction now requires that the four assumptions (A1), (A2), (A3$^+$) and (A4) of reference [5] hold for the measure $\delta_\varLambda$. Let us fix an averaging sequence $\mathcal{A}$ of van Hove type; the result will not depend on this choice, due to the unique ergodicity of $(\mathbb{X}(\varLambda), G)$.

As $\varLambda$ is Meyer, the measure $\delta_\varLambda = \sum_{x\in\varLambda} \delta_x$ is translation bounded, i.e., for all compact $K \subset G$, there exists a constant C_K with $\sup_{t\in G} \delta_\varLambda(t+K) \le C_K$. This is just the validity of (A1) for the measure $\delta_\varLambda$.

As $(\mathbb{X}(\varLambda), G)$ is uniquely ergodic, the autocorrelation

$$\gamma \; := \; \lim_{n\to\infty} \frac{1}{\theta_G(A_n)} \sum_{x,y\in\varGamma\cap A_n} \delta_{x-y} \tag{8}$$

exists for every $\varGamma \in \mathbb{X}(\varLambda)$, does not depend on $\varGamma$, and equals $\sum_{x\in\varDelta} \eta(x)\delta_x$, with $\varDelta = \varLambda - \varLambda$ and a suitable positive definite function $\eta\colon G \longrightarrow \mathbb{C}$. This is assumption (A2) for $\delta_\varLambda$.

Note that $\eta(0) = \mathrm{dens}(\varLambda)$, and $\eta(x) = 0$ whenever $x \notin \varLambda - \varLambda$. In fact, the function η is closely connected to the metric d described above in (1) and (2). More precisely, a direct calculation gives

$$d(s+\varLambda, t+\varLambda) \; := \; \lim_{n\to\infty} \frac{\mathrm{card}\left(\left((s+\varLambda) \bigtriangleup (t+\varLambda)\right) \cap A_n\right)}{\theta_G(A_n)} \; = \; 2\big(\eta(0) - \eta(t-s)\big)\,. \tag{9}$$

The set $\{x \in G : \eta(x) \neq 0\}$ is clearly a subset of $\varDelta$ and hence uniformly discrete, as $\varLambda$ is Meyer, and this is assumption (A3$^+$).

Finally, as $\beta_{\mathbb{A}}$ is continuous, its image $\mathbb{A}(\varLambda)$ is compact. By [28], this implies (see Lemma 5 below as well), that $\widehat{\gamma}$ is a pure point measure on $\widehat{G}$. This in turn means that, for each $\varepsilon > 0$, the set of ε-almost periods defined in (4) is relatively dense in G, compare [5]. This is assumption (A4).

We close this section by noting that the ε-almost periods do not depend on $\varLambda$, but only on $\mathbb{X}(\varLambda)$. More precisely, by uniform existence of the autocorrelation (8) and (9), for every $\varGamma \in \mathbb{X}(\varLambda)$, the identities

$$P_\varepsilon = \{t \in \varLambda - \varLambda : d_G(t,0) < \varepsilon\} \; = \; \{t \in G : d_G(t,0) < \varepsilon\} \; = \; \{t \in \varGamma - \varGamma : d(t,0) < \varepsilon\} \tag{10}$$

hold whenever $\varepsilon < 2\eta(0)$.

4.2. **Creating a cut and project scheme.** Here, we use the method of [5] to construct a CPS out of γ and $\varLambda$. This is possible since we have just established the validity of the necessary conditions (A1), (A2), (A3$^+$) and (A4).

Let L be the group generated by the set $\varDelta = \varLambda - \varLambda$. Clearly, the pseudo-metric d discussed in (2) restricts to L and gives a pseudo-metric d_L by

$$d_L(s,t) \;:=\; d_G(s,t) \;=\; d(s+\varLambda, t+\varLambda) \;=\; 2\big(\eta(0)-\eta(t-s)\big)\,, \tag{11}$$

where the last equality follows from (9). The topology on L defined by this is again called the *autocorrelation topology*. It makes L into a topological group.

A fundamental system of neighbourhoods of 0 in L is given by the P_ε, $\varepsilon > 0$, defined above in Eq. (4). Let H be the Hausdorff completion of L under the autocorrelation topology and let $\phi:\; L \longrightarrow H$ be the corresponding completion map. It should be noted that ϕ is not injective in general. In fact, if $\varLambda$ is a lattice, one finds $H = \{0\}$.

In any case, let $\mathcal{L}$ be the subgroup $\{(t,\phi(t)) \mid t \in L\}$. Then, this subgroup is a lattice in $G\times H$ and we arrive at a CPS $(G,H,\mathcal{L})$ as shown in (6). The pseudo-metric d_L on L induces a corresponding metric d_H on H. Let B^H_ε denote the corresponding open ball of radius ε in H. Then,

$$P_\varepsilon \;=\; \phi^{-1}\big(\phi(L)\cap B^H_\varepsilon\big). \tag{12}$$

Proposition 4. *Let $\varLambda \subset G$ be a Meyer set such that $(\mathbb{X}(\varLambda), G)$ is uniquely ergodic. Then, $\varDelta = \varLambda - \varLambda$ is totally bounded (or precompact) in the autocorrelation topology. In particular, $\overline{\phi(\varDelta)}$ and $\overline{\phi(\varLambda)}$ are compact subsets of H.*

Proof. The subsets P_ε, $0<\varepsilon<2\eta(0)$, form a fundamental system of neighbourhoods for 0 in L. Fix one of them. It is relatively dense in G and hence there is a compact K with $G = P_\varepsilon + K$. Let $s \in \varDelta$ and write $s = t+k$, with $t \in P_\varepsilon$ and $k \in K$. Then, $s-t \in (\varDelta-\varDelta)\cap K$ which is a finite set F since $\varDelta$ is a Meyer set (so, $\varDelta-\varDelta$ is uniformly discrete, see the Appendix). Finally, $s = t+s-t \in P_\varepsilon + F$, so $\varDelta \subset P_\varepsilon + F$, showing that $\varDelta$ is totally bounded. □

Let $\mathbb{T} = \mathbb{T}(\varLambda) := (G\times H)/\mathcal{L}$ be the corresponding compact Abelian quotient group. There is a natural action of G on $\mathbb{T}$, defined by letting $x \in G$ act as $(u,v)+\mathcal{L} \mapsto (x+u,v)+\mathcal{L} \in \mathbb{T}$ for all $(u,v) \in G\times H$. This way, $\mathbb{T}$ becomes a dynamical system for G, both measure theoretically (using the Haar measure $\theta_{\mathbb{T}}$) and topologically. The G-orbit of $0 \in \mathbb{T}$ is dense in $\mathbb{T}$, as is every other orbit. The homomorphism $\iota\colon\; G \longrightarrow \mathbb{T}$ provided by this orbit is not injective in general: its kernel is $\ker(\phi) \subset L$, the set of *statistical periods* of $\varLambda$. Clearly, ϕ plays the role of the $\star$-map, wherefore we once again write $t^\star$ rather than $\phi(t)$ from now on.

Now, the important fact is that the compact group $\mathbb{T}$ we have just constructed agrees with $\mathbb{A}(\varLambda)$ defined in Section 2.3. More precisely, we have the following result from [28], which follows from the definition of $\mathbb{A}(\varLambda)$ and the characterization of $\mathbb{T}$ as the completion of G in the so-called mixed topology given in [5]. For the convenience of the reader, we sketch a proof.

Proposition 5. *Let $\varLambda \subset G$ be a Meyer set such that $(\mathbb{X}(\varLambda), G)$ is uniquely ergodic, and let $\beta_{\mathbb{A}}:\; \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$ be the corresponding torus parametrization. Then, $\mathbb{T} \simeq \mathbb{A}(\varLambda)$, and this isomorphism is a G-map when both spaces are given their natural G-actions.*

Proof. Let $\alpha: L \longrightarrow G \times L$ be the diagonal map. Then, $\alpha(L)$ is discrete in $G \times L$ and $(G \times L)/\alpha(L)$ becomes a topological group in the usual way. Furthermore, $G \simeq (G \times L)/\alpha(L)$ via the canonical embedding $x \mapsto (x, 0) + \alpha(L)$, and we provide G with a new topology this way, called the *mixed topology*. There is a homomorphism of $(G \times L)/\alpha(L)$ into the compact group $\mathbb{T} = (G \times H)/\mathcal{L}$ defined by $(x, t) + \alpha(L) \mapsto (x, t^\star) + \mathcal{L}$. In [5], it is shown that, via this map, $\mathbb{T}$ is the Hausdorff completion of $(G \times L)/\alpha(L)$. Therefore, $\mathbb{T}$ may be identified with the Hausdorff completion of G in the mixed topology and $\iota(G) \subset \mathbb{T}$ is the Hausdorff space *associated* with G. Given this construction of $\mathbb{T}$, we are left with the task to relate the mixed topology to the autocorrelation topology.

By definition, a basis for the open neighbourhoods of 0 in G, in the mixed topology, consists of the sets of the form $V+P_\varepsilon$, V an open neighbourhood of 0 in the original topology of G, $\varepsilon > 0$ (as these are precisely the sets in G which correspond to the sets $V \times P_\varepsilon + \alpha(L) \subset (G \times L)/\alpha(L)$ under our isomorphism). On the other hand, as discussed in Section 2, the autocorrelation completion $\mathbb{A}$ of G comes about by supplying G with the uniformity induced from $\mathcal{D}$ which has the sets $U_{\mathrm{mACT}}(V, \varepsilon) = \{(\varLambda', \varLambda'') : \exists v \in V \text{ with } d(v + \varLambda', \varLambda'') < \varepsilon\}$. The corresponding neighbourhoods of 0 in G are then

$$U_{\mathrm{mACT}}(V, \varepsilon)(0) \;=\; \{x \in G : \exists v \in V \text{ with } d(v + \varLambda, x + \varLambda) < \varepsilon\}\,.$$

Now, the definition of P_ε implies

$$U_{\mathrm{mACT}}(V, \varepsilon)(0) \;=\; V + P_\varepsilon\,,$$

and the proof is complete. □

The key consequence of Proposition 5 is that our map $\beta_{\mathbb{A}}: \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$ can be interpreted as a continuous G map $\beta_{\mathbb{T}}: \mathbb{X}(\varLambda) \longrightarrow \mathbb{T}$. This gives

Theorem 2. *Let $\varLambda$ be a Meyer set for which $(\mathbb{X}(\varLambda), G)$ is uniquely ergodic. Suppose that there exists a continuous G map $\beta_{\mathbb{A}}: \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$. Then, there is a CPS $(G, H, \mathcal{L})$ with associated compact Abelian group $\mathbb{T}$ for which $\mathbb{A} \simeq \mathbb{T}$ via a topological isomorphism which is a G-map that sends $\varLambda \in \mathbb{A}$ to $0 \in \mathbb{T}$. In particular, there is a torus parametrization $\beta_{\mathbb{T}}: \mathbb{X}(\varLambda) \longrightarrow \mathbb{T}$.* □

5. Consequences of the existence of non-singular elements: The window

We continue to assume that $\varLambda$ is Meyer such that the associated dynamical system $(\mathbb{X}(\varLambda), G)$ is uniquely ergodic and that there exists a torus parametrization, i.e., a continuous G-map $\beta_{\mathbb{A}}: \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$. In this section, we investigate some consequences, first that $\beta_{\mathbb{A}}$ is non-singular at least at one element, and second that $\beta_{\mathbb{A}}$ is non-singular almost everywhere.

5.1. **Existence of a non-singular element.** Assume that $\mathbb{X}(\varLambda)$ has at least one non-singular element, see Definition 4. Thus, we have a dynamical subsystem $\mathbb{X}(\varLambda)_R$ that is the closure of the set of non-singular elements $R(\mathbb{X})$ of $\mathbb{X}(\varLambda)$, as defined in Proposition 2.

In the previous section, we have constructed a CPS from $\varLambda$ as well as a continuous map $\beta_{\mathbb{T}}: \mathbb{X}(\varLambda) \longrightarrow \mathbb{T}$. In this section, we aim at

Theorem 3. *Let $\varLambda \subset G$ be a Meyer set such that $(\mathbb{X}(\varLambda), G)$ is uniquely ergodic. Assume that there exists a continuous G-map $\beta_{\mathbb{A}} : \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$, which is one-to-one at least at one point. Then, there is an irredundant CPS $(G, H, \mathcal{L})$ associated with $\mathbb{X}(\varLambda)$ and a subset $W \subset H$, $W = \overline{W^\circ}$ compact, so that every non-singular element of $\mathbb{X}(\varLambda)$ is of the form*

$$\varGamma \; = \; x + \curlywedge(-h + W^\circ) \; = \; x + \curlywedge(-h + W)$$

for some $(x, h) \in G \times H$.

Each element of $\mathbb{X}(\varLambda)_R$ is repetitive and an IMS for the window W. If $\varLambda$ itself is repetitive, one has $\mathbb{X}(\varLambda) = \mathbb{X}(\varLambda)_R$.

The proof requires some preparation. The following lemma is one of the cornerstones of the present work. It says that the mixed autocorrelation topology, which is defined by statistical information at infinity, is actually compatible with the local topology, which is defined by local information, whenever a certain condition is met. This condition is that $\varGamma$ is non-singular relative to $\beta_{\mathbb{A}}$. As mentioned above, we always assume in this section that $\varLambda \subset G$ is a Meyer set such that $(\mathbb{X}(\varLambda), G)$ is uniquely ergodic.

Lemma 3. *Let $\mathcal{A}$ be an averaging sequence for G as introduced above, and let $\varGamma \in \mathbb{X}(\varLambda)$ be non-singular. Given any positive integer M, there is an $\varepsilon = \varepsilon(M) > 0$ so that*

$$t \in P_\varepsilon \quad \Longrightarrow \quad (t + \varGamma) \cap A_M \; = \; \varGamma \cap A_M \, .$$

Proof. By Proposition 5, $\mathbb{T} \simeq \mathbb{A}(\varLambda)$. Now, the statement can be concluded from Lemma 2 after noticing that $d(\beta_{\mathbb{A}}(t + \varGamma), \beta_{\mathbb{A}}(\varGamma)) < \varepsilon$ whenever $t \in P_\varepsilon$. Namely, Lemma 2 then implies that $t + \varGamma$ and $\varGamma$ are arbitrarily close in the local topology if ε is sufficiently small. As $\varGamma$ is Meyer and $P_\varepsilon \subset \varGamma - \varGamma$ by (10), Fact 4 implies that $\varGamma$ and $t + \varGamma$ actually agree on arbitrarily large compact sets, such as A_M, if $\varepsilon > 0$ is chosen accordingly. □

As a consequence of Lemma 3, and extending an argument used before in [6], we can show that every non-singular element of $\mathbb{X}(\varLambda)$ is a model set:

Proposition 6. *If $\varGamma$ is a non-singular element of $\mathbb{X}(\varLambda)$ with $0 \in \varGamma$, one has $\varGamma = \curlywedge(W^\circ) = \curlywedge(W)$, where $W := \overline{\varGamma^\star}$ and $W = \overline{W^\circ}$.*

Proof. By $0 \in \varGamma$, we have $\varGamma \subset \varGamma - \varGamma \subset \varLambda - \varLambda \subset L$. Now, let $x_0 \in \varGamma$. Choose a positive integer M so that $x_0 \in A_M$. Choose $\varepsilon(M)$ according to Lemma 3.

Let $y \in L$ and suppose that $y^\star \in x_0^\star + B^H_{\varepsilon(M)}$. Then, $y^\star - x_0^\star \in L^\star \cap B^H_{\varepsilon(M)} = P^\star_{\varepsilon(M)}$, which implies $y - x_0 \in P_{\varepsilon(M)}$. Then, $x_0 - y \in P_{\varepsilon(M)}$ and, by Lemma 3, $(x_0 - y + \varGamma) \cap A_M = \varGamma \cap A_M$. This implies $x_0 - y + u = x_0$ for some $u \in \varGamma$. Then, $y = u \in \varGamma$, so $\varGamma \supset \curlywedge(x_0^\star + B^H_{\varepsilon(M)})$ and $x_0^\star + B^H_{\varepsilon(M)} \subset W$. This shows that

$$\curlywedge(x_0^\star + B^H_{\varepsilon(x_0)}) \; \subset \; \varGamma \, , \quad \text{for all } x_0 \in \varGamma, \tag{13}$$

where $\varepsilon(x_0)$ is the $\varepsilon(M)$ of the previous lemma. Now,

$$W := \overline{\varGamma^\star} = \overline{\bigcup_{x_0 \in \varGamma} (x_0^\star + B^H_{\varepsilon(x_0)})} \; \supset \; \bigcup_{x_0 \in \varGamma} (x_0^\star + B^H_{\varepsilon(x_0)}) =: V.$$

Obviously, V is open and contains $\varGamma^\star$. Thus, $\overline{W^\circ} \supset \overline{V} \supset \overline{\varGamma^\star} \; = \; W$ and $W = \overline{V} = \overline{W^\circ}$.

By (13), $\varGamma = \curlywedge(V)$. As $\varGamma$ belongs to $\mathbb{X}(\varLambda)$, a restriction gives a continuous torus parametrization

$$\beta_{\mathbb{T}}|_{\varGamma} :\; \mathbb{X}(\varGamma) \subset \mathbb{X}(\varLambda) \;\longrightarrow\; \mathbb{T}.$$

As $\varGamma$ is non-singular, the torus parametrization $\beta_{\mathbb{T}}$ and then even more the torus parametrization $\beta_{\mathbb{T}}|_{\varGamma}$ is one-to-one at $\varGamma$.

We next show $\partial V \cap L^{\star} = \varnothing$. If $p \in G$ satisfies $p^{\star} \in \partial V \cap L^{\star}$, then, by denseness of $L^{\star}$, we can find a net $(t_i) \in L$ with $(t_i^{\star})$ in $V \cap L^{\star}$ and $t_i^{\star} \to p^{\star}$. Without loss of generality, we may assume that $p - t_i + \varGamma = \curlywedge(p^{\star} - t_i^{\star} + V)$ converges to some element $\varGamma' \in \mathbb{X}(\varGamma)$. Then, $\varGamma \neq \varGamma'$ as one contains p and the other does not. On the other hand, for some $(a,b) \in G \times H$,

$$\beta_{\mathbb{T}}|_{\varGamma}(\varGamma) \;=\; (a,b) + L^{\star} \;=\; \lim_i\,(a, b + p^{\star} - t_i^{\star}) + L^{\star} \;=\; \lim_i \beta_{\mathbb{T}}|_{\varGamma}(p - t_i + \varGamma) \;=\; \beta_{\mathbb{T}}|_{\varGamma}(\varGamma')\,,$$

contradicting the non-singularity of $\varGamma$.

By $\partial V \cap L^{\star} = \varnothing$, we have

$$\varGamma \;=\; \curlywedge(V) \;=\; \curlywedge(V \cup \partial V) \;=\; \curlywedge(\overline{V}) \;=\; \curlywedge(W)\,.$$

As $\varGamma = \curlywedge(V)$ and $V \subset W^{\circ}$, we infer $\varGamma = \curlywedge(W^{\circ})$ as well, and the proof is complete. □

We can use Proposition 6 to show that the CPS we have just created is irredundant, and also to determine that each element in the orbit closure of the non-singular elements, i.e., in $X(\varLambda)_R$, is an IMS for some translate of the same window W.

Proposition 7. *Let a CPS $(G, H, \mathcal{L})$ be given, together with a window $W \subset H$ that is non-empty, compact, and satisfies $W = \overline{W^{\circ}}$. Consider an IMS $\varLambda$ with $\curlywedge(W^{\circ}) \subset \varLambda \subset \curlywedge(W)$. With $\mathbb{T} = (G \times H)/\mathcal{L}$ as above, the following assertions are equivalent.*

(i) *There exists a continuous G-map $\beta_{\mathbb{T}} \colon \mathbb{X}(\varLambda) \longrightarrow \mathbb{T}$ with $\beta_{\mathbb{T}}(\varLambda) = (0,0) + \mathcal{L}$*
(ii) *The window W is irredundant, i.e., $W = c + W$ implies $c = 0$.*

In this case, $\varLambda' \in \mathbb{X}(\varLambda)$ satisfies $\beta_{\mathbb{T}}(\varLambda') = (x,h) + \mathcal{L}$ if and only if $x + \curlywedge(-h + W^{\circ}) \subset \varLambda' \subset x + \curlywedge(-h + W)$ holds.

Proof. The implication (ii) $\Longrightarrow$ (i) follows by the argument given in [36] to prove the case $\varLambda = \curlywedge(W)$ (see [19] as well).

The implication (i) $\Longrightarrow$ (ii) and the last statement will be proved together. This will be done in three steps. To this end, let $\beta_{\mathbb{T}} \colon \mathbb{X}(\varLambda) \longrightarrow \mathbb{T}$ be continuous with $\beta_{\mathbb{T}}(\varLambda) = (0,0) + \mathcal{L}$, and consider an arbitrary $\varLambda' \in \mathbb{X}(\varLambda)$.

Step 1: *$\beta_{\mathbb{T}}(\varLambda') = (x,h) + \mathcal{L}$ implies $x + \curlywedge(-h + W^{\circ}) \subset \varLambda' \subset x + \curlywedge(-h + W)$.*

Let (x,h) be given with $\beta_{\mathbb{T}}(\varLambda') = (x,h) + \mathcal{L}$, and let $y \in G$ be chosen so that $0 \in \varLambda'' := -y + \varLambda'$. Let $\{t_n + \varLambda\}_n$, $t_n \in G$, be a net converging to $\varLambda''$ in $\mathbb{X}(\varLambda)$. Without loss of generality, we may assume that $0 \in t_n + \varLambda$ for all n. Then, in particular, $t_n \in -\varLambda$ and therefore $t_m^{\star} - t_n^{\star} \in W - W$ for all n, m. As $W - W$ is compact, we may assume that $\{t_n^{\star}\}_n \to -k \in H$, possibly after restricting to a subnet.

Now, $\beta_{\mathbb{T}}(t_n + \varLambda) = \iota(t_n) + \beta_{\mathbb{T}}(\varLambda)$, where, since $\beta_{\mathbb{T}}$ is a G-map, $\iota(t_n) = (t_n, 0) + \mathcal{L} = (0, -t_n^{\star}) + \mathcal{L}$, which converges to $(0,k) + \mathcal{L}$ in $\mathbb{T}$. Thus, by continuity of $\beta_{\mathbb{T}}$, $\beta_{\mathbb{T}}(\varLambda'') = (0,k) + \mathcal{L}$

and $\beta_{\mathbb{T}}(\varLambda') = \beta_{\mathbb{T}}(y + \varLambda'') = (y, k) + \mathcal{L}$. As, by assumption, $\beta_{\mathbb{T}}(\varLambda') = (x, h) + \mathcal{L}$, we infer $(y, k) + \mathcal{L} = (x, h) + \mathcal{L}$. This gives

$$(14)\quad x + \curlywedge(-h + W^{\circ}) \;=\; y + \curlywedge(-k + W^{\circ}) \quad\text{and}\quad x + \curlywedge(-h + W) \;=\; y + \curlywedge(-k + W).$$

Consider an arbitrary $z \in \curlywedge(-k + W^{\circ})$, so that $z^{\star} + k \in W^{\circ}$. Then, for all large n, $z^{\star} - t_n^{\star} \in W^{\circ}$ and

$$z \in \curlywedge(t_n^{\star} + W^{\circ}) \;=\; t_n + \curlywedge(W^{\circ}) \;\subset\; t_n + \varLambda.$$

Thus, $z \in \varLambda''$ and $\curlywedge(-k + W^{\circ}) \subset \varLambda''$ follows. Adding y, and invoking (14), we end up with

$$x + \curlywedge(-h + W^{\circ}) \;\subset\; \varLambda'.$$

Conversely, if $z \in \varLambda''$, then $z \in t_n + \varLambda$ for sufficiently large n, so that $z^{\star} - t_n^{\star} \in W$ and, in the limit, $z^{\star} \in -k + W$, i.e., $z \in \curlywedge(-k + W)$, which implies $\varLambda' \subset y + \curlywedge(-k + W)$. Again, using (14), we obtain

$$\varLambda' \;\subset\; x + \curlywedge(-h + W).$$

STEP 2: $c + W = W$ *implies* $c = 0$*, i.e., condition (ii) holds.*
Note that $c + W = W$ implies $c + W^{\circ} = W^{\circ}$. As $W = \overline{W^{\circ}}$, the boundary of W is nowhere dense. By the Baire category theorem, there exists then a $d \in H$ with

$$\curlywedge(d + W^{\circ}) \;=\; \curlywedge(d + W).$$

Moreover, $\beta_{\mathbb{T}}$ is onto by Lemma 1. Thus, there exist $\varLambda', \varLambda'' \in \mathbb{X}(\varLambda)$ with

$$(15)\qquad \beta_{\mathbb{T}}(\varLambda') \;=\; (0, -d) + \mathcal{L}\,, \quad \beta_{\mathbb{T}}(\varLambda'') \;=\; (0, -d - c) + \mathcal{L}\,.$$

By the result of Step 1, this implies

$$\curlywedge(d + W^{\circ}) \;\subset\; \varLambda' \;\subset\; \curlywedge(d + W) \quad\text{as well as}\quad \curlywedge(d + c + W^{\circ}) \;\subset\; \varLambda'' \;\subset\; \curlywedge(d + c + W)\,.$$

By our choice of d, and because we both have $c + W = W$ and $c + W^{\circ} = W^{\circ}$, we can infer $\varLambda' = \varLambda''$. This, in turn, implies $\beta_{\mathbb{T}}(\varLambda') = \beta_{\mathbb{T}}(\varLambda'')$, and $c = 0$ follows from (15).
STEP 3: $x + \curlywedge(-h + W^{\circ}) \subset \varLambda' \subset x + \curlywedge(-h + W)$ *implies* $\beta_{\mathbb{T}}(\varLambda') = (x, h) + \mathcal{L}$*.*
Let (y, f) with $\beta_{\mathbb{T}}(\varLambda') = (y, f) + \mathcal{L}$ be given. By Step 1, we then have

$$y + \curlywedge(-f + W^{\circ}) \;\subset\; \varLambda' \;\subset\; y + \curlywedge(-f + W)\,.$$

Adding $-x$ yields

$$(16)\qquad y - x + \curlywedge(-f + W^{\circ}) \;\subset\; \varLambda' - x \;\subset\; y - x + \curlywedge(-f + W)\,.$$

On the other hand, the assumption on (x, h) gives

$$(17)\qquad \curlywedge(-h + W^{\circ}) \;\subset\; \varLambda' - x \;\subset\; \curlywedge(-h + W)\,.$$

These inclusions show that $(y - x)$ belongs to L and we can rewrite (16) as

$$(18)\qquad \curlywedge((y - x)^{\star} - f + W^{\circ}) \;\subset\; \varLambda' - x \;\subset\; \curlywedge((y - x)^{\star} - f + W)\,.$$

Now, a combination of (17) and (18) gives

$$\curlywedge((y - x)^{\star} - f + W^{\circ}) \;\subset\; \curlywedge(-h + W) \quad\text{and}\quad \curlywedge(-h + W^{\circ}) \;\subset\; \curlywedge((y - x)^{\star} - f + W)\,,$$

which in turn implies

$$((y - x)^{\star} - f + W^{\circ}) \cap L^{\star} \;\subset\; -h + W \quad\text{and}\quad (-h + W^{\circ}) \cap L^{\star} \;\subset\; (y - x)^{\star} - f + W\,.$$

Taking closures and using $\overline{W^\circ} = W$ as well as the denseness of $L^\star$ in H, we obtain

$$(y-x)^\star - f + W \subset -h + W \quad \text{and} \quad -h + W \subset (y-x)^\star - f + W.$$

These inclusions yield $f - h - (y-x)^\star + W = W$ and, by Step 2,

$$f - h - (y-x)^\star \;=\; 0.$$

This, however, means $(y,f) + \mathcal{L} = (x,h) + \mathcal{L} = \beta_{\mathbb{T}}(\varLambda')$, and the proof of Step 3, and also of the entire claim, is complete. □

5.2. **The proof of Theorem 3.** To prove Theorem 3, consider a non-singular element $\varGamma$ of $\mathbb{X}(\varLambda)$. By translating $\varGamma$, we may assume $0 \in \varGamma$ without loss of generality. Proposition 6 then implies $\varGamma = \curlywedge(W^\circ) = \curlywedge(W)$, where $W := \overline{\varGamma^\star}$ and $W = \overline{W^\circ}$ is compact. By Proposition 5, $\mathbb{A}(\varLambda) \simeq \mathbb{T}$.

Assume $\varGamma = \varLambda$ for the moment. Then, by Proposition 7, every $\varLambda' \in \mathbb{X}(\varLambda)$ is an IMS of the form that we require. If, on the other hand, $\varLambda$ is singular, these results apply to all the elements of $\mathbb{X}(\varLambda)_R$, since it contains all the non-singular elements and is the closed hull of any of its elements. As pointed out in Fact 2, the elements of $\mathbb{X}(\varLambda)_R$ are all repetitive.

This finishes the proof of Theorem 3. □

Remark 1. There is very little that one can say about the generator $\varLambda$ of the hull $\mathbb{X}(\varLambda)$ being a model set, or even an IMS, if repetitivity or some other consistency property is not assumed. One can, for instance, take a model set, add some finite set of spurious points, and take the hull of the resulting set. That destroys the set as a model set, but does not destroy the properties of the minimal part $\mathbb{X}(\varLambda)_R$ of the hull, which will not have been altered. However, with the assumption of non-singularity almost everywhere, we can obtain information up to sets of density 0.

5.3. **Consequences of non-singularity almost everywhere.** In this section, we shall prove Theorem 6. To do so, we need some preparation around the regularity of the window in the cut and project scheme. To do so, we assume the following setting.

- **(S)** $(G, H, \mathcal{L})$ is a CPS, $W \subset H$ is a non-empty, compact set with $W = \overline{W^\circ}$, and $\varLambda$ is an arbitrary IMS for it, i.e., $\curlywedge(W^\circ) \subset \varLambda \subset \curlywedge(W)$. There exists a continuous G-map $\beta_{\mathbb{T}} \colon \mathbb{X}(\varLambda) \longrightarrow \mathbb{T}$ with $\beta_{\mathbb{T}}(\varLambda) = (0,0) + \mathcal{L}$.

Proposition 7 has the following consequence.

Proposition 8. *Let* **(S)** *be valid, with an IMS $\varLambda$. For any $c \in H$, the following properties are equivalent.*

- (i) $\curlywedge(-c + W^\circ) = \curlywedge(-c + W)$*;*
- (ii) $\partial(-c + W) \cap L^\star = \varnothing$*;*
- (iii) *The fibre over* $(0,c) + \mathcal{L}$ *is non-singular.*

In this case, $\curlywedge(-c+W^\circ) = \curlywedge(-c+W)$ *constitutes the fibre over* $(0,c)+\mathcal{L}$*, and one has the inclusion* $\mathbb{X}(\curlywedge(-c+W)) \subset \mathbb{X}(\varLambda)$*.*

Proof. The equivalence of (i) and (ii) is obvious. Also, **(S)** allows us to use Proposition 7, whence we see that (i) implies (iii).

It remains to show that (iii) implies (ii), or its contraposition. To this end, let us assume that there is some $p \in L$ with $p^\star \in \partial(-c+W)$. The $\beta_{\mathbb{T}}$-fibre over $(0,c)+\mathcal{L}$ is non-empty and consists of the elements $\varLambda' \in \mathbb{X}(\varLambda)$ such that

$$\curlywedge(-c+W^\circ) \subset \varLambda' \subset \curlywedge(-c+W)$$

by Proposition 7. We claim that there are at least two elements on this fibre, one of which contains p while the other does not.

Take any $\varLambda'$ on the fibre. Suppose first that $p \notin \varLambda'$. Since $p^\star$ is on the boundary of $-c+W$, there is a net $\{\ell_n\}$ in L with $\{\ell_n^\star\} \longrightarrow c$, such that $p \in \curlywedge(-\ell_n^\star + W^\circ)$ for all n. Then, on the fibre over $(0,\ell_n^\star)+\mathcal{L}$, there is a set $\varLambda_n \in \mathbb{X}(\varLambda)$ with $\curlywedge(-\ell_n^\star + W^\circ) \subset \varLambda_n \subset \curlywedge(-\ell_n^\star + W)$. By the compactness of $\mathbb{X}(\varLambda)$, there is a convergent subnet of $\{\varLambda_n\}$ which we may assume to be $\{\varLambda_n\}$ itself. Let $\{\varLambda_n\} \longrightarrow \varLambda'' \in \mathbb{X}(\varLambda)$. Then, $p \in \varLambda_n$ for all n implies $p \in \varLambda''$. Also, $\beta_{\mathbb{T}}(\varLambda'') = \lim_n \beta_{\mathbb{T}}(\varLambda_n) = \lim_n (0,\ell_n^\star)+\mathcal{L} = (0,c)+\mathcal{L}$, so $\varLambda''$ is on the same fibre as $\varLambda'$, but it contains p whereas $\varLambda'$ does not.

The argument for the case when $p \in \varLambda'$ is similar. This time, choose a net $\{\ell_n\}$ in L with $\{\ell_n^\star\} \to c$, $p \notin \curlywedge(-\ell_n^\star + W)$, for all n. We then find $\varLambda_n$ on the fibre over $(0,\ell_n^\star)+\mathcal{L}$, with $p \notin \varLambda_n$, and get $\varLambda'' \in \mathbb{X}(\varLambda)$ on the fibre over $(0,c)+\tilde{L}$, also with $p \notin \varLambda''$.

The last statement of the Proposition is obvious. $\square$

Next, let us relate the properties of W versus ∂W to the injectivity of $\beta_{\mathbb{T}}$.

Theorem 4. [25] *Let $(G,H,\mathcal{L})$ be a CPS. Let M be a measurable, relatively compact set in H. Then,*

$$\operatorname{dens}(x+\curlywedge(M-h)) \;:=\; \lim_{n\to\infty} \frac{\operatorname{card}\big((x+\curlywedge(M-h))\cap A_n\big)}{\theta_G(A_n)} \;=\; \operatorname{dens}(\mathcal{L})\,\theta_H(M),$$

which is valid for all $(x,h) \in G\times H$ if $\theta_H(\partial M)=0$, and otherwise for $\theta_G\times\theta_H$-almost every $(x,h)\in G\times H$. $\square$

Lemma 4. *Let $M \subset \mathbb{T}$ be any measurable subset whose preimage in $G\times H$ is contained in a subset of the form $G\times B$ with $\theta_H(B)=0$. Then, $\theta_{\mathbb{T}}(M)=0$.*

Proof. Observe first that $\theta_{G\times H}(A\times B) = \theta_G(A)\,\theta_H(B) = 0$ for any relatively compact measurable set $A \subset G$. Since G is σ-compact, we may now employ the averaging sequence $\mathcal{A}=\{A_n\}$ of Section 2.1, with $A_n \subset A_{n+1}$ and $G=\bigcup_n A_n$, to conclude that also $\theta_{G\times H}(G\times B)=0$.

Let now $\pi\colon\, G\times H \longrightarrow \mathbb{T}$ be the canonical projection. Define, for $\xi\in\mathbb{T}$, the measure ν_ξ on $G\times H$ by

$$\nu_\xi \;:=\; \sum_{z\in\pi^{-1}(\xi)} \delta_z\,.$$

Standard desintegration (e.g., using a fundamental domain) shows that $\theta_{G\times H} = \theta_{\mathbb{T}}\circ\nu$, i.e., $\int f(z)\,\mathrm{d}\theta_{G\times H}(z) = \int \nu_\xi(f)\,\mathrm{d}\theta_{\mathbb{T}}(\xi)$ for any measurable nonnegative f on $G\times H$. This gives

$$0 \le \theta_{\mathbb{T}}(M) = \int 1_M(\xi)\,\mathrm{d}\theta_{\mathbb{T}}(\xi) \le \int \nu_\xi(1_M\circ\pi)\,\mathrm{d}\theta_{\mathbb{T}}(\xi) \le \int \nu_\xi(1_{G\times B})\,\mathrm{d}\theta_{\mathbb{T}}(\xi) = \theta_{G\times H}(G\times B)$$

and the proof is finished because the last term vanishes as shown above. $\square$

Theorem 5. *Let* **(S)** *be in place. Then, the boundary of W has measure 0 if and only if $\beta_{\mathbb{T}}$ is one-to-one almost everywhere.*

Proof. By Proposition 8, $\varLambda' \in \mathbb{X}(\varLambda)$ is non-singular if and only if $\varLambda' = x + \curlywedge(-h + W^\circ) = x + \curlywedge(-h + W)$ and $L^\star \cap (-h + \partial W) = \varnothing$ for some $(x,h) \in G \times H$. In this case, one has $x + \curlywedge(-h + \partial W) = \varnothing$. We thus have $\mathrm{dens}(x + \curlywedge(-h + \partial W)) = 0$ at this point.

If $\beta_{\mathbb{T}} : \mathbb{X}(\varLambda) \longrightarrow \mathbb{T}$ is one-to-one $\mathbb{T}$-a.e., we also have this relation $G \times H$-a.e., due to $\theta_{G\times H} = \theta_{\mathbb{T}} \circ \nu$ (see the proof of the previous lemma). Consequently, by Theorem 4 and because $\mathrm{dens}(\mathcal{L}) \neq 0$, we may conclude that $\theta_H(\partial W) = 0$.

Conversely, suppose that $\theta_H(\partial W) = 0$. By Proposition 8,

$$\begin{aligned} F &:= \{\xi \in \mathbb{T} : \text{the fibre over } \xi \text{ contains more than one element}\} \\ &= \{(x,h) + \mathcal{L} \in \mathbb{T} : \curlywedge(-h + W^\circ) \neq \curlywedge(-h + W)\}\,. \end{aligned}$$

This gives

$$\begin{aligned} \theta_{\mathbb{T}}(F) &= \theta_{\mathbb{T}}\big(\{(x,h) + \mathcal{L} \in \mathbb{T} : \curlywedge(-h + W^\circ) \neq \curlywedge(-h + W)\}\big) \\ &= \theta_{\mathbb{T}}\big(\{(x,h) + \mathcal{L} \in \mathbb{T} : h \in L^\star + \partial W\}\big) \\ &= \theta_{\mathbb{T}}\big(G \times (L^\star + \partial W) \bmod \mathcal{L}\big) \;=\; 0, \end{aligned}$$

where we used Lemma 4 in the last step together with the fact that $L^\star$ is countable. □

We can now proceed to the final result of this section.

Theorem 6. *Let G be a σ-compact LCA group and $\varLambda$ a Meyer subset of G such that the canonical map $\beta : \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$ is continuous and one-to-one almost everywhere, with respect to the Haar measure on $\mathbb{A}(\varLambda) = \beta(\mathbb{X}(\varLambda))$. Then, $\mathbb{X}(\varLambda)$ is uniquely ergodic and $\varLambda$ agrees with a regular model set up to a set of density 0. Furthermore, if $\varLambda$ is repetitive, $\mathbb{X}(\varLambda)$ is actually associated to a regular model set.*

Proof. We are given a Meyer set $\varLambda$ and assume that $\beta\colon \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$ is continuous and one-to-one almost everywhere. We first want to show that $\varLambda$ differs from a model set up to a set of points of density 0. As β is continuous, $\mathbb{A}(\varLambda)$ is compact. Moreover, G acts minimally on $\mathbb{A}(\varLambda)$ by definition. Thus, $(\mathbb{A}(\varLambda), G)$ is uniquely ergodic with the Haar measure $\theta_{\mathbb{A}}$ on $\mathbb{A}(\varLambda)$. We show that $(\mathbb{X}(\varLambda), G)$ is uniquely ergodic as well.

As β is one-to-one almost everywhere, there exists a subset $\mathbb{A}' \subset \mathbb{A}(\varLambda)$ of full measure such that β is one-to-one on $\mathbb{X}' := \beta^{-1}(\mathbb{A}')$ and the complement of $\mathbb{X}'$ is mapped into the complement of $\mathbb{A}'$ by β. Now, note that, by Proposition 3, any inverse of β is continuous when restricted to $\mathbb{A}'$. Thus, extending this continuous function, say by setting it constant on $\mathbb{A}(\varLambda) \setminus \mathbb{A}'$, we find a measurable $\alpha\colon \mathbb{A} \longrightarrow \mathbb{X}(\varLambda)$, which is an inverse to β on $\mathbb{A}'$.

Let μ be any G-invariant probability measure on $\mathbb{X}(\varLambda)$. As $(\mathbb{A}(\varLambda), G)$ is uniquely ergodic, $\beta^*(\mu)$ is the Haar measure $\theta_{\mathbb{A}}$ on $\mathbb{A}(\varLambda)$. In particular, $\mu(\beta^{-1}(M)) = 0$ whenever M is a subset of $\mathbb{A}(\varLambda)$ of measure 0. In particular, $\mu(\mathbb{X}(\varLambda) \setminus \mathbb{X}') = 0$. Let f be any measurable bounded function on $\mathbb{X}(\varLambda)$. Then, f and $f \circ \alpha \circ \beta$ only differ on $\beta^{-1}(\mathbb{A}(\varLambda) \setminus \mathbb{A}')$, which has μ-measure 0. This implies

$$\mu(f) \;=\; \mu(f \circ \alpha \circ \beta) \;=\; \beta^*(\mu)(f \circ \alpha) \;=\; \theta_{\mathbb{A}}(f \circ \alpha) \;=\; \alpha^*(\theta_{\mathbb{A}})(f)\,.$$

Thus, μ is uniquely determined and the unique ergodicity of $(\mathbb{X}(\varLambda), G)$ follows.

Now, the assumptions of Theorems 2 and 3 are satisfied, and we find both a CPS such that $\mathbb{A}(\varLambda) \simeq \mathbb{T}$ and a dynamical system $(\mathbb{X}(\varLambda)_R, G)$ associated to a model set $\varGamma = \curlywedge(W)$ inside of $(\mathbb{X}(\varLambda), G)$ with irredundant W and (metrizable) internal group H.

From the previous results, and Theorem 5 in particular, we know that the almost one-to-one-ness of β forces the boundary of W to have measure 0. Consider the fibre lying over $(x, h) + \mathcal{L}$. If the fibre is non-singular, the single element of $\mathbb{X}(\varLambda)$ is $x + \curlywedge(h + W)$, which is a *regular* model set. Even if the fibre is singular, set(s) lying there differ by density 0 from the regular model set $x + \curlywedge(h + W)$, since $\mathrm{dens}(x + \curlywedge(h + \partial W)) = 0$ by Theorem 4.

Of course, if $\varLambda$ is repetitive, $\mathbb{X}(\varLambda)$ is generated by any of its elements, and so $\mathbb{X}(\varLambda)$ is actually associated to a regular model set. □

6. Existence of a continuous $\beta_{\mathbb{A}}$ and pure point spectrum with continuous eigenfunctions

Let $\varLambda$ be a Meyer set such that $(\mathbb{X}(\varLambda), G)$ is a uniquely ergodic dynamical system. The existence of a torus parametrization $\beta_{\mathbb{A}}\colon \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$ has proved to be the key to linking $\varLambda$ to the realm of model sets. In this section, we connect the existence of a torus parametrization with properties of the dynamical system $\mathbb{X}(\varLambda)$ itself. These properties are pure pointedness of the spectrum and continuity of the eigenfunctions.

Theorem 7. *Let $\varLambda$ be a Meyer subset of G such that $(\mathbb{X}(\varLambda), G)$ is uniquely ergodic. Then, the following assertions are equivalent.*

(a) *There exists a continuous G-map $\beta_{\mathbb{A}}\colon \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$.*

(b) *$(\mathbb{X}(\varLambda), G)$ has pure point dynamical spectrum with continuous eigenfunctions.*

In this case, $\varGamma, \varGamma' \in \mathbb{X}(\varLambda)$ satisfy $\beta_{\mathbb{A}}(\varGamma) = \beta_{\mathbb{A}}(\varGamma')$ if and only if $f(\varGamma) = f(\varGamma')$ for every eigenfunction f.

This and the next section of the paper are devoted to the proof of this result. In this section, we prove Theorem 7 in the direction (a) $\Rightarrow$ (b). In the following section, we prove the converse.

6.1. The proof of (a) $\Longrightarrow$ (b) of Theorem 7.

Let $\varLambda$ be a Meyer set with associated uniquely ergodic dynamical system $(\mathbb{X}(\varLambda), G, \mu)$ and let T be the corresponding unitary representation of G on $L^2(\mathbb{X}(\varLambda), \mu)$.

Recall that the eigenvalues of this dynamical system form a subgroup of $\widehat{G}$, which we denote by $P(T)$. In addition, we need to consider the diffraction measure $\widehat{\gamma}$, which is constant on $\mathbb{X}(\varLambda)$ due to the unique ergodicity. For any measure ν on $\widehat{G}$, we introduce the set

$$P(\nu) := \{k \in \widehat{G} : \nu(\{k\}) \neq 0\}, \tag{19}$$

which is a countable set, and the subgroup of $\widehat{G}$ that it generates, denoted by $\langle P(\nu)\rangle$.

We recall the following result that has already been established in the literature.

Lemma 5. *Let $\varLambda \subset G$ be a Meyer set. If $(\mathbb{X}(\varLambda), G)$ is uniquely ergodic, the following assertions are equivalent.*

(i) *$\mathbb{A}(\varLambda)$ is compact.*

(ii) *$\widehat{\gamma}$ is a pure point measure.*
(iii) *$(\mathbb{X}(\varLambda), G)$ has pure point dynamical spectrum.*

In this case, the dynamical spectrum $P(T)$ of $(\mathbb{X}(\varLambda), G)$ satisfies $P(T) = \langle P(\widehat{\gamma})\rangle$.

Proof. The equivalence of (i) and (ii) is shown in [28]. The equivalence of (ii) and (iii) is proved in [20, Thm. 3.2]. The last statement is proved in [2, Thm. 9]. □

If there exists a continuous G-map $\beta_{\mathbb{A}} : \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$, Theorem 2 tells us that we have a CPS $(G, H, \mathcal{L})$ and a compact group $\mathbb{T} = (G \times H)/\mathcal{L}$. Moreover, $\mathbb{A}(\varLambda) = \mathbb{T}$. Thus $\beta_{\mathbb{A}}$ induces a continuous map $\beta_{\mathbb{T}}$ between $\mathbb{X}(\varLambda)$ and $\mathbb{T}$. There is then a canonical homomorphism $\iota \colon G \longrightarrow \mathbb{T}$ of topological groups with dense range defined by $x \mapsto (x, 0) + \mathcal{L}$. Dualizing, we obtain an injective homomorphism $\hat{\iota} \colon \widehat{\mathbb{T}} \longrightarrow \widehat{G}$ of the dual topological groups. Lemma 5 tells us that $\widehat{\gamma}$ is a pure point measure.

Lemma 6. *Let $\varLambda$ be a Meyer set such that $(\mathbb{X}(\varLambda), G)$ is uniquely ergodic and $\widehat{\gamma}$ is a pure point measure. Then, $\langle P(\widehat{\gamma})\rangle \subset \hat{\iota}(\widehat{\mathbb{T}})$.*

Proof. Due to unique ergodicity, each element of $\mathbb{X}(\varLambda)$ has the same autocorrelation measure γ. Let $C_{\mathrm{c}}(G)$ denote the space of continuous complex-valued functions of compact support on G. For every $c \in C_{\mathrm{c}}(G)$, we define $g_c : G \longrightarrow \mathbb{C}$ by $g_c = c * \tilde{c} * \gamma$. Then, there is a continuous positive definite function $g_c^{\mathbb{T}}$ on $\mathbb{T}$ so that $g_c^{\mathbb{T}} \circ \iota = g_c$ (see Section 4 of [5] as well). In particular, we can expand $g_c^{\mathbb{T}}$ in a uniformly converging Fourier series

$$g_c^{\mathbb{T}}(x) \; = \; \sum_{k \in \widehat{\mathbb{T}}} a_c(k)(k, x)$$

with nonnegative numbers $a_c(k)$ that satisfy

$$\sum_{k \in \widehat{\mathbb{T}}} a_c(k) \; = \; g_c(0)\, .$$

Composing $g_c^{\mathbb{T}}$ with the homomorphism $\iota \colon G \longrightarrow \mathbb{T}$ and using the definition of $\hat{\iota}$, we obtain

$$g_c(x) \; = \; \sum_{k \in \widehat{\mathbb{T}}} a_c(k)\, (\hat{\iota}(k), x)\, .$$

As the $a_c(k)$ are summable, we can calculate the Fourier transform of g_c to arrive at

$$|\widehat{c}|^2 \widehat{\gamma} \; = \; \widehat{g_c} \; = \; \sum_{k \in \widehat{\mathbb{T}}} a_c(k)\, \delta_{\hat{\iota}(k)}\, ,$$

which is a finite positive measure on $\widehat{G}$.

This shows that

$$B := \{k \in \widehat{\mathbb{T}} : a_c(k) > 0 \text{ for some continuous } c \text{ with compact support}\}$$

is mapped unter $\hat{\iota}$ into $P(\widehat{\gamma})$ defined in (19). Taking for c an approximate unit, one infers that B is actually mapped onto $P(\widehat{\gamma})$. As $\hat{\iota}(\widehat{\mathbb{T}})$ is a subgroup of $\widehat{G}$, which then contains $\iota(B)$, the desired conclusion follows immediately. □

Proposition 9. *Let $\varLambda$ be a Meyer set in G such that $(\mathbb{X}(\varLambda), G)$ is uniquely ergodic. If there exists a continuous G-mapping $\beta_{\mathbb{A}} \colon\ \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$, then $(\mathbb{X}(\varLambda), G)$ has pure point dynamical spectrum with continuous eigenfunctions.*

Proof. By Lemma 5, the dynamical system $(\mathbb{X}(\varLambda), G)$ has pure point dynamical spectrum. Moreover, as discussed after the lemma, we can then identify $\mathbb{A}(\varLambda)$ and $\mathbb{T}$. Thus, $\beta_{\mathbb{A}}$ yields a continuous G-map from $\mathbb{X}(\varLambda)$ to $\mathbb{T}$.

Every element $\lambda \in \widehat{\mathbb{T}}$ gives rise to a continuous eigenfunction

$$f_\lambda := \lambda \circ \beta_{\mathbb{T}} \colon\ \mathbb{X}(\varLambda) \longrightarrow \mathbb{C}$$

to the eigenvalue $\hat{\imath}(\lambda)$, and we infer

$$\hat{\imath}(\widehat{\mathbb{T}}) \subset P(T)\,,$$

where the point spectrum $P(T)$ is the set of eigenvalues. Combining this with the results of Lemma 5 and Lemma 6, we obtain the following chain of inclusions:

$$\hat{\imath}(\widehat{\mathbb{T}}) \ \subset\ P(T) = \langle P(\widehat{\gamma})\rangle \ \subset\ \hat{\imath}(\widehat{\mathbb{T}})\,.$$

Therefore, $\hat{\imath}(\widehat{\mathbb{T}}) = P(T)$. Thus, the f_λ, $\lambda \in \widehat{\mathbb{T}}$, provide eigenfunctions for each eigenvalue. As each eigenvalue has multiplicity one by ergodicity, we have found a complete system of eigenfunctions, all of which are continuous. □

This finishes the proof of Theorem 7 in the direction $(a) \Rightarrow (b)$. □

Remark 2. Under the hypotheses of Proposition 9, $\langle P(\widehat{\gamma})\rangle = \hat{\imath}(\widehat{\mathbb{T}})$. This also holds under the assumptions of Lemma 6 (and, in fact, in much more general situations as well). This will be discussed further in [4].

7. Consequences of unique ergodicity, pure point dynamical spectrum, and continuous eigenfunctions

The aim of this section is to prove the following theorem. As discussed at the end of this section, this theorem will provide the proof of the missing direction of Theorem 7.

Theorem 8. *Let $\varLambda \subset G$ be a Meyer set and $(\mathbb{X}(\varLambda), G)$ be uniquely ergodic. Suppose that $(\mathbb{X}(\varLambda), G)$ has pure point dynamical spectrum and continuous eigenfunctions. Let $\mathbb{A}(\varLambda)$ be the autocorrelation hull. Then, there exists a torus parametrization from $\mathbb{X}$ to $\mathbb{A}$ for which $\varLambda \mapsto 0$.*

To prove this result, we proceed as follows. In Paragraph 7.1, we assume that $(\mathbb{X}, G)$ is an arbitrary uniquely ergodic dynamical system with pure point spectrum and continuous eigenfunctions. We then show how to construct a compact topological group $\mathbb{E}$ and a continuous surjective G-map $\beta_{\mathbb{E}} \colon\ \mathbb{X} \longrightarrow \mathbb{E}$. In the subsequent paragraphs, we return to the case of $(\mathbb{X}, G)$ being a Meyer dynamical system, assuming now that we have pure point spectrum and continuous eigenfunctions, and show that the continuous map $\beta_{\mathbb{E}} \colon\ \mathbb{X} \longrightarrow \mathbb{E}$ constructed in the first paragraph is effectively none other than a torus parametrization $\beta_{\mathbb{A}} \colon\ \mathbb{X} \longrightarrow \mathbb{A}$.

Remark 3. As investigated by Robinson [33] in the case of $G = \mathbb{R}^d$ and $G = \mathbb{Z}^d$, continuity of the eigenfunctions is related to uniform existence of certain limits (see [22] for recent results in the case of general LCA groups G as well).

7.1. **A general construction.** Let $(\mathbb{X}, G)$ be a uniquely ergodic dynamical system with unique G-invariant probability measure μ. This gives rise to a unitary representation T of G on $L^2(\mathbb{X}, \mu)$.

Assume that T has pure point dynamical spectrum with all eigenfunctions continuous. This means that $L^2(\mathbb{X}, \mu)$ has an orthonormal basis $\{f_\lambda : \lambda \in P(T)\}$ where the point spectrum $P(T)$ (i.e., the set of eigenvalues of T) is some *subgroup* of $\widehat{G}$, the character group of G, and each f_λ is a continuous eigenfunction for the character λ. Note that, due to ergodicity, all eigenvalues are simple, and the corresponding eigenspaces are thus one-dimensional [40]. We may assume that each f_λ is normalized to 1 (in the L^2-norm).

Define $\varLambda' \sim \varLambda''$ when $f_\lambda(\varLambda') = f_\lambda(\varLambda'')$ for all $\lambda \in P(T)$. Let $\mathbb{E} := \mathbb{X}/\!\sim$ and let $\beta_{\mathbb{E}}$ denote the canonical mapping from $\mathbb{X}$ to $\mathbb{E}$. Note that the f_λ can be factored through the equivalence relation. Give the quotient space the uniform structure for which the cylinder sets given by

$$U(F,\varepsilon) \;:=\; \{(\beta_{\mathbb{E}}(\varLambda'), \beta_{\mathbb{E}}(\varLambda'')) \in \mathbb{E}\times\mathbb{E} : |f_\lambda(\varLambda') - f_\lambda(\varLambda'')| < \varepsilon, \lambda \in F\}\,,$$

where F runs through all finite subsets of $P(T)$ and ε through the positive reals, are a fundamental system of entourages. The mapping $\beta_{\mathbb{E}} : \mathbb{X} \longrightarrow \mathbb{E}$ is uniformly continuous because the eigenfunctions are continuous (hence uniformly continuous, since $\mathbb{X}$ is compact). Thus, $\mathbb{E}$ is compact and hence complete.

Each of the basic entourages of $\mathbb{E}$ is actually G-invariant (since the f_λ are *eigenfunctions*) and we obtain from this an obvious G-action on $\mathbb{E}$ for which the natural mapping $\beta_{\mathbb{E}}$ from $\mathbb{X}$ to $\mathbb{E}$ is a G-map. This implies the orbit $\beta_{\mathbb{E}}(G+\varLambda)$ of $\beta_{\mathbb{E}}(\varLambda)$ to be dense in $\mathbb{E}$.

Pull back the uniformity of $\mathbb{E}$ to G by using the entourages

$$\big\{(s,t) \in G\times G : \big(\beta_{\mathbb{E}}(s+\varLambda), \beta_{\mathbb{E}}(t+\varLambda)\big) \in U(F,\varepsilon)\big\}.$$

This new uniformity on G is compatible with the group structure (this comes down again to the G-invariance of each of the fundamental entourages) and we have a uniformly continuous mapping of G, equipped with this new topology, into $\mathbb{E}$. In fact, $\mathbb{E}$ is the completion of G under this new uniform topology, and since this latter is also an Abelian group, $\mathbb{E}$ becomes a compact Abelian group. In more detail, since G is getting its structure by pulling back the induced structure on $\beta_{\mathbb{E}}(G+\varLambda)$ in $\mathbb{E}$ under $t \mapsto \beta_{\mathbb{E}}(t+\varLambda)$, we may apply [9, Ch. II.3.8, Prop. 17] to see that the Hausdorff space associated with G under this new uniformity is homeomorphic to $\beta_{\mathbb{E}}(G+\varLambda)$ and hence also their completions are homeomorphic.

So, out of the continuity of the eigenfunctions, we obtain a new compact Abelian group $\mathbb{E}$ and a torus parametrization

$$\beta_{\mathbb{E}} \colon\; \mathbb{X} \longrightarrow \mathbb{E}\,. \tag{20}$$

By construction, we have:

Proposition 10. *For each $\lambda \in P(T)$, there exists a unique continuous function g_λ on $\mathbb{E}$ with $f_\lambda = g_\lambda \circ \beta_{\mathbb{E}}$.* □

Since $(\mathbb{E}, G, \theta_{\mathbb{E}})$ is pure point with eigenvalues $P(T)$, it must be conjugate to the G-action on $\mathbb{S} := \widehat{P(T)}$. This, and in fact a more general statement, is known as the Halmos–von Neumann representation theorem, compare [40, Thm. 5.18]. In the case at hand, we give a short proof. Explicitly, equip the subgroup $P(T)$ with the discrete topology, so that its dual

group $\mathbb{S}$ is compact. Since G is mapped homomorphically into $\mathbb{S}$ (namely each $g \in G$ goes to the evaluation map at g of $P(T)$), it follows that $\mathbb{S}$ admits a canonical (and minimal) action of G. Fix $x_0 \in \mathbb{E}$ and normalize the g_λ by requiring $g_\lambda(x_0) = 1$, $\lambda \in P(T)$. Then, $|g_\lambda(x)| = 1$ for all $x \in \mathbb{E}$.

Proposition 11. *The groups $\mathbb{E}$ and $\mathbb{S}$ are isomorphic as topological groups by the mapping $j : \mathbb{E} \longrightarrow \mathbb{S}$ defined by $j(x)\colon\ P(T) \longrightarrow U(1),\ \lambda \mapsto g_\lambda(x)$, and thereby the dynamical systems $(\mathbb{E}, G)$ and $(\mathbb{S}, G)$ are topologically conjugate.*

Proof. By the normalization condition on $g_\lambda(x_0)$, we infer

$$g_\lambda(x) g_\mu(x) \;=\; g_{\lambda\mu}(x) \quad \text{and} \quad g_{\lambda^{-1}}(x) \;=\; \overline{g_\lambda(x)}.$$

This implies that $j(x)$ is indeed an element of $\mathbb{S} = \widehat{P(T)}$. Now, continuity of j follows directly from the continuity of the g_λ. One checks that j is a G-map. Injectivity of j follows as the g_λ, $\lambda \in P(T)$, separate the points of $\mathbb{E}$. To show that j is onto, it suffices to show that the dual j^* of j

$$j^*\colon\ \widehat{\mathbb{S}} = P(T) \longrightarrow \widehat{\mathbb{E}}, \quad j^*(\lambda) := \lambda \circ j$$

is injective (since the image of j is closed and the action of G on $\mathbb{S}$ is minimal). This can be seen as follows. Let $\lambda_1, \lambda_2 \in P(T)$ be given, with

$$j^*(\lambda_1) \;=\; j^*(\lambda_2).$$

Then, $\lambda_1 \circ j = \lambda_2 \circ j$, i.e., $j(x)(\lambda_1) = j(x)(\lambda_2)$ for every $x \in \mathbb{E}$. As $j(x)(\lambda_1) = g_{\lambda_1}(x)$ and, similarly, for λ_2, we see that this means $g_{\lambda_1} = g_{\lambda_2}$ which, in turn, implies $f_{\lambda_1} = f_{\lambda_2}$, and finally $\lambda_1 = \lambda_2$.

Thus, we see that j is indeed a continuous bijection between compact spaces. Therefore, the inverse of j is continuous as well.

One has to show two more things: that j is compatible with the group action, and that j is a group homomorphism. Both of them are more or less straightforward calculations. $\square$

7.2. **Pure point dynamical spectrum together with continuous eigenfunctions imply a torus parametrization.** In this section, we specialize the setting of the previous paragraph by assuming that $\varLambda$ is Meyer and $(\mathbb{X}(\varLambda), G)$ is uniquely ergodic with pure point dynamical spectrum and continuous eigenfunctions. The main objective is to prove Theorem 8. Given equation (20), it remains to show that $\mathbb{E}$ as constructed above is isomorphic to the hull $\mathbb{A} = \mathbb{A}(\varLambda)$ of $\varLambda$ in the mixed autocorrelation topology. This is done in the next two paragraphs. Here, we provide some preparation.

As in the proof of Lemma 6, let $C_{\mathrm{c}}(G)$ denote the space of continuous complex-valued functions of compact support on G and define, for $c \in C_{\mathrm{c}}(G)$, the function $\varphi_c\colon\ \mathbb{X} \longrightarrow \mathbb{C}$ by $\varphi_c(\varLambda) = \big(c * \delta_\varLambda\big)(0)$. Let $g_c := c * \tilde{c} * \gamma_\varLambda$ be the corresponding smoothed out autocorrelation of $\varLambda$, which is a continuous function on G. From unique ergodicity and Dworkin's argument [11, 20],

$$g_c(t) \;=\; \langle T_t \varphi_c, \varphi_c \rangle \tag{21}$$

where $t \in G$ and $\langle .,. \rangle$ is the inner product on $L^2(\mathbb{X}, \mu)$ whereby it is a Hilbert space. It is not hard to see that the function $t \mapsto \langle T_t f, f \rangle$ is continuous, bounded and positive definite

for any $f \in L^2(\mathbb{X},\mu)$. Thus, by Bochner's theorem [23], there exists a unique finite positive measure σ_f on $\widehat{G}$ with

$$\langle T_t f, f\rangle \;=\; \int_{\widehat{G}} (\widehat{s},t)\,\mathrm{d}\sigma_f(\widehat{s})$$

(see [2, 3] as well for a further discussion of this). It turns out that, in our context, the spectral measure σ_{φ_c} can be explicitly calculated for $c \in C_{\mathrm{c}}(G)$ in terms of $\widehat{\gamma}$. More precisely,

$$\sigma_{\varphi_c} \;=\; |\widehat{c}|^2 \widehat{\gamma}. \tag{22}$$

for any $c \in C_{\mathrm{c}}(G)$, compare [36, 20, 2]. This equation links the dynamical spectrum and the diffraction spectrum.

Proposition 12. *Let λ be an eigenvalue of the uniquely ergodic dynamical system $(\mathbb{X}(\varLambda),G)$, with associated normalized eigenfunction f_λ. Let $h \in L^2(\mathbb{X}(\varLambda),\mu)$ be arbitrary. Then, $\langle h, f_\lambda\rangle = 0$ if and only if $\sigma_h(\{\lambda\}) = 0$.*

Proof. By Stone's theorem, compare [23, Sec. 36D], there exists a projection valued measure

$$E\colon \text{ Borel sets of } \widehat{G} \;\longrightarrow\; \text{projections on } L^2(\mathbb{X}(\varLambda),\mu)$$

with $E(B\cap C) = E(B)E(C)$ for $B,C \subset \widehat{G}$ measurable and

$$\langle T_t f, g\rangle \;=\; \int_{\widehat{G}} (\hat{s},t)\,\mathrm{d}\sigma_{f,g}(\hat{s}) \tag{23}$$

where the measure $\sigma_{f,g}$ on $\widehat{G}$ is defined by $\sigma_{f,g}(B) := \langle E(B)f, g\rangle$. In particular, we have $\sigma_{f,f} = \sigma_f$.

From $E(B)E(\{\lambda\}) = E(B\cap\{\lambda\})$, we infer that $\sigma_{E(\{\lambda\})f,g}$ is concentrated on $\{\lambda\}$ for arbitrary $f,g \in L^2(\mathbb{X}(\varLambda),\mu)$. This easily yields that $T_t E(\{\lambda\})f = (\lambda,t)E(\{\lambda\})f$ for any $f \in L^2(\mathbb{X}(\varLambda),\mu)$.

Conversely, if f is an eigenfunction to λ, we infer from the validity of the equation $(\lambda,t)\langle f,g\rangle = \langle T_t f, g\rangle = \int_{\widehat{G}} (\hat{s},t)\,\mathrm{d}\sigma_{f,g}(\hat{s})$, for all $t \in G$, that $\sigma_{f,g}$ is concentrated on λ. This easily gives $E(\{\lambda\})f = f$ for any eigenfunction to the eigenvalue λ.

Put together, this means that $E(\{\lambda\})$ is the orthogonal projection onto the eigenspace for the eigenvalue λ. This eigenspace is one-dimensional by ergodicity. Thus, $E(\{\lambda\})h = \langle h, f_\lambda\rangle f_\lambda$ and we infer

$$|\langle h, f_\lambda\rangle|^2 \;=\; \|E(\{\lambda\})h\|^2 \;=\; \langle E(\{\lambda\})h, E(\{\lambda\})h\rangle \;=\; \sigma_h(\{\lambda\}).$$

Now, the statement of the proposition is immediate. □

We are now ready to prove the isomorphism between $\mathbb{E}$ and $\mathbb{A} = \mathbb{A}(\varLambda)$. Both spaces in question, $\mathbb{E}$ and $\mathbb{A}$, are obtained by completion of G in uniform topologies for which a fundamental system of G-*invariant* entourages exist. For this reason, it is actually sufficient to show that the identity mapping from G to itself is bi-continuous at 0 when these two topologies are put on two sides.

7.3. **Continuity of $\mathbb{A} \longrightarrow \mathbb{E}$.** If $\varLambda$ is a Meyer set, we know that $\varDelta = \varLambda - \varLambda$ is uniformly discrete. Consequently, there is a compact neighbourhood K of 0 in G so that, for all $t \in \varDelta$, $(t + K) \cap \varDelta = \{t\}$.

Let $\{x_i\}$ be a net in G which converges to 0 in the autocorrelation topology. Then, there are elements $v_i \in G$ converging to 0 in the original topology of G so that $d(v_i + x_i + \varLambda, \varLambda)$ converges to 0. Here, d is the pseudo-metric defined in (1), which, by (9), satisfies

$$d(s+\varLambda, t+\varLambda) \;=\; \lim_{n\to\infty} \frac{\operatorname{card}\big(\big((s+\varLambda)\;\triangle\;(t+\varLambda)\big)\cap A_n\big)}{\theta_G(A_n)} \;=\; 2\big(\eta(0) - \eta(s-t)\big)\,,$$

so, for $y_i := v_i + x_i$,

$$d(y_i + \varLambda, \varLambda) \;=\; 2\big(\eta(0) - \eta(y_i)\big) \;\to\; 0\,. \tag{24}$$

This convergence of the $\{y_i + \varLambda\}$ to $\varLambda$ shows that $y_i \in \varDelta$ for all sufficiently large i. If we show that $\{y_i\}$ converges to 0 in the topology of $\mathbb{E}$, this will also give convergence of the original net $\{x_i\}$, since the topology of $\mathbb{X}(\varLambda)$, and hence $\mathbb{E}$, is defined so that shifts by small elements of G are small.

Now, $\gamma_\varLambda = \sum_{t\in\varDelta} \eta(t)\delta_t$, Let $c \in C_c(G)$ with $\operatorname{supp}(c * \tilde{c}) \subset K$. By our choice of K, 0 is then the only element of $\varDelta$ in $\operatorname{supp}(c * \tilde{c})$. Thus,

$$g_c(y_i) \;=\; \big(c * \tilde{c} * \gamma_{\varLambda}\big)(y_i) \;=\; \sum_{t\in\varDelta} \eta(t)\,\big(c*\tilde{c}\big)(y_i - t) \;=\; \eta(y_i)\,\big(c * \tilde{c}\big)(0)\,.$$

By (24), this implies $\lim_i g_c(y_i) = g_c(0)$. By (21), this means that

$$\langle T_{y_i}\varphi_c, \varphi_c\rangle \;\longrightarrow\; \langle \varphi_c, \varphi_c\rangle. \tag{25}$$

As we have pure point spectrum with the set of eigenvalues $P(T)$ and corresponding normalized eigenfunctions f_λ, $\lambda \in P(T)$, we can write φ_c as a Fourier series, $\varphi_c = \sum_{\lambda\in P(T)} a_\lambda f_\lambda$, where the a_λ (which depend on c) are complex numbers.

Then, using (25), we find $\sum \lambda(y_i)\,|a_\lambda|^2 \|f_\lambda\|^2 \longrightarrow \sum |a_\lambda|^2 \|f_\lambda\|^2$ which results in

$$\sum |a_\lambda|^2 \big(\lambda(y_i) - 1\big) \;\longrightarrow\; 0\,,$$

by normalization of the eigenfunctions. Taking complex conjugates then yields

$$\sum |a_\lambda|^2 (\overline{\lambda(y_i)} - 1) \;\longrightarrow\; 0\,.$$

As λ takes values in $U(1)$, we have

$$|\lambda(y_i) - 1|^2 \;=\; (1 - \lambda(y_i)) + (1 - \overline{\lambda(y_i)})$$

and we obtain

$$\sum |a_\lambda|^2 |\lambda(y_i) - 1|^2 \;=\; \sum |a_\lambda|^2 \big(1 - \lambda(y_i) + 1 - \overline{\lambda(y_i)}\big) \;\longrightarrow\; 0\,.$$

Thus, $\{\lambda(y_i)\} \to 1$, whenever $a_\lambda \neq 0$. Now, $a_\lambda \neq 0$ means that φ_c is not orthogonal to f_λ. By Proposition 12, this is equivalent to to $\lambda \in P(\sigma_{\varphi_c})$ (see (19) for the definition of $P(\cdot)$). Thus, we have $\{\lambda(y_i)\} \to 1$ for all $\lambda \in P(\sigma_{\varphi_c})$ and for all $c \in C_c(G)$. As $c \in C_c(G)$ is arbitrary, (22) then shows that $\{\lambda(y_i)\} \to 1$ for all $\lambda \in P(\widehat{\gamma})$, and then for all $\lambda \in \langle P(\widehat{\gamma})\rangle$. As $P(\widehat{\gamma}) = P(T)$ by Lemma 5, this means, for all $\lambda \in P(T)$,

$$f_\lambda(y_i + \varLambda) \;=\; \lambda(y_i) f_\lambda(\varLambda) \;\longrightarrow\; f_\lambda(\varLambda)$$

and this is precisely the meaning of convergence of $\{y_i\}$ to 0 in the $\mathbb{E}(\varLambda)$-topology. This concludes the continuity argument in the first direction.

7.4. **Continuity of $\mathbb{E} \longrightarrow \mathbb{A}$.** Let $\lambda \in P(T)$, so $f_\lambda(x+\varLambda) = \lambda(x) f_\lambda(\varLambda)$ for all $x \in G$. The continuity of f_λ then shows that $|f_\lambda|$ is a non-zero constant function on $\mathbb{X}(\varLambda)$. Let $\{x_i\} \to 0$ in the $\mathbb{E}$-topology on G. Then, $\{f_\lambda(x_i+\varLambda)\} \to f_\lambda(\varLambda)$ shows that $\{\lambda(x_i)\} \to 1$ for all $\lambda \in P(T)$.

Let $c \in C_{\mathrm{c}}(G)$ be chosen so that $0 \le c(x) \le 1$ for all $x \in G$, with $c(x) = 1 \Leftrightarrow x = 0$. Moreover, let c be so that $\nu := c * \tilde{c}$ satisfies $(\mathrm{supp}(\nu) - \mathrm{supp}(\nu)) \cap (\varDelta - \varDelta) = \{0\}$, which rests upon the Meyer property. Then, $\|c\|_2^2 = \nu(0) > \nu(x) \ge 0$ for all $x \in G \backslash \{0\}$ and $\mathrm{supp}(\nu) \cap \varDelta = \{0\}$.

Let $\varphi_c = \sum_{\lambda \in P(T)} a_\lambda f_\lambda$ be the Fourier expansion of φ_c. Choose $\varepsilon > 0$ and find a finite set $F \subset P(T)$ so that

$$\|\varphi_c - \sum_{\lambda \in F} a_\lambda f_\lambda\|_2 \;<\; \varepsilon\,.$$

Choose N in the index set of $\{x_i\}$ so that, for all $i \succcurlyeq N$ and all $\lambda \in F$,

$$|\lambda(x_i) - 1| \;<\; \frac{\varepsilon}{1 + \sum_{\lambda \in F} |a_\lambda|}\,.$$

Then,

$$\begin{aligned} \|T_{x_i}\varphi_c - \varphi_c\|_2 \;&<\; \bigl\|T_{x_i}\varphi_c - T_{x_i}\sum_{\lambda\in F} a_\lambda f_\lambda\bigr\|_2 \\ &+\; \bigl\|\sum_{\lambda\in F}\lambda(x_i)\,a_\lambda f_\lambda - \sum_{\lambda\in F} a_\lambda f_\lambda\bigr\|_2 + \bigl\|\sum_{\lambda\in F} a_\lambda f_\lambda - \varphi_c\bigr\|_2 \\ &<\; \varepsilon + \sum_{\lambda\in F} |\lambda(x_i) - 1|\,|a_\lambda|\,\|f_\lambda\|_2 + \varepsilon \;<\; 3\varepsilon\,, \end{aligned}$$

since the T_x are unitary and $\|f_\lambda\|_2 = 1$.

Thus, $\{T_{x_i}\varphi_c\} \to \varphi_c$ and hence $\{\langle T_{x_i}\varphi_c, \varphi_c\rangle\} \to \langle \varphi_c, \varphi_c\rangle$ and, using (21) again,

$$\bigl\{g_c(x_i) = \sum_{t\in\varDelta} \nu(x_i - t)\eta(t)\bigr\} \;\longrightarrow\; g_c(0)\,. \tag{26}$$

For each x_i, there is at most one $t_i \in \varDelta$ with $x_i - t_i \in \mathrm{supp}(\nu)$. Thus,

$$g_c(x_i) \;=\; \begin{cases} \nu(x_i - t_i)\eta(t_i), & \text{if } t_i \text{ exists,} \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, $g_c(0) = \nu(0)\eta(0)$.

Now, (26) implies that $\nu(x_i - t_i)\eta(t_i) \to \nu(0)\eta(0) \neq 0$ (so, in particular, the t_i must exist eventually). Since $0 \le \nu(x_i - t_i) \le \nu(0)$ and $0 \le \eta(t_i) \le \eta(0)$, we get $\{\eta(t_i)\} \to \eta(0)$ and $\{\nu(x_i - t_i)\} \to \nu(0)$. By the choice of c, $\{v_i := t_i - x_i\} \to 0$.

Now, $\{x_i\}$ converges to 0 in the $\mathbb{A}$-topology since $\{v_i + x_i\} = \{t_i\}$, the $\{v_i\} \to 0$ in the original topology of G, and $\{d(t_i + \varLambda, \varLambda) = 2(\eta(0) - \eta(t_i))\} \to 0$.

This implies continuity in the other direction and completes the proof of Theorem 8. □

7.5. **Proof of Theorem 7, (b) $\Longrightarrow$ (a).** This is immediate from Theorem 8. □

8. The proof of the sufficiency direction of Theorem 1

Proof. The hypotheses of Theorem 1, in the direction of sufficiency, include those of Theorem 7. Thus, we have a torus parametrization $\mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$ coming from the mapping $\mathbb{X}(\varLambda) \longrightarrow \mathbb{E}$. This provides us with a cut and project scheme according to Theorem 2.

By assumption, our torus parametrization is one-to-one almost everywhere. Thus, Theorem 3 and Theorem 5 apply. Therefore, we obtain a window by Theorem 3, whose boundary has Haar measure 0 in H by Theorem 5.

Repetitivity of $\varLambda$ is equivalent to the minimality of $\mathbb{X}(\varLambda)$, which is assumed. Thus, $\mathbb{X}(\varLambda)$ is associated with a regular model set according to Theorem 3 as required. □

9. The proof of necessity in Theorem 1

Up to now, the direction of investigation has been from dynamical systems and torus parametrizations to cut and project schemes and model sets. In this section, we derive results going in the other direction and prove that the conditions of Theorem 1 are necessary.

9.1. **Irredundancy.** Assume that we are given an IMS $\varLambda$ with respect to the CPS $(G, H, \mathcal{L})$. We wish to construct a torus parametrization for the local hull of $\varLambda$. As we have seen in Proposition 7, the curious property of irredundancy is crucial to the existence of such a map. What happens if we have a CPS together with an IMS for which the window fails irredundancy? Can we modify the CPS (and thereby the torus) to get the irredundancy? The answer is yes. But it is here that the notion of an *inter* model set becomes important.

Lemma 7. *Let $(G, H, \mathcal{L})$ be a CPS. Let $W \subset H$ be a non-empty compact set with $\overline{W^\circ} = W$ and $\theta_H(\partial W) = 0$. Let $\varLambda \subset G$ with $⅄(W^\circ) \subset \varLambda \subset ⅄(W)$ be arbitrary. Then, there exists a CPS $(G, H', \mathcal{L}')$ and $W' \subset H'$ compact, non-empty and irredundant, with $W' = \overline{W'^\circ}$ and $\theta_{H'}(\partial W') = 0$, such that $⅄(W'^\circ) \subset \varLambda \subset ⅄(W')$, i.e., that $\varLambda$ is a regular IMS for the new CPS $(G, H', \mathcal{L}')$ with window W'.*

Remark 4. The proof of the lemma relies on factoring out the stabilizer of W. It is crucial to note that sets of the form $⅄(W)$ may *not* be representable as sets of the form $⅄(W')$ in the emerging "quotient" scheme. Rather, sets lying between $⅄(W^\circ)$ and $⅄(W)$ can be exhibited as sets lying between $⅄(W'^\circ)$ and $⅄(W')$. We refer the reader to [19] for further discussion.

Proof. Let $(G, H, \mathcal{L})$ be the given CPS, with the usual conditions on the projections π_1 and π_2, and with the compact regular window $W = \overline{W^\circ} \neq \varnothing$. Let

$$I \;:=\; \mathrm{stab}_H(W) \;=\; \{t \in H : t + W = W\}$$

be the stabilizer of W, which is a subgroup of H. Clearly, I is closed, and $I \subset W - W$ implies that I is compact. Observe that we also have $W^\circ + I = W^\circ$.

Define the factor group $H' = H/I$ and let $\rho\colon\; H \longrightarrow H'$ be the natural map. Moreover, define

$$\mathcal{L}' \;:=\; \{(x, \rho(x^\star)) : x \in L\} \;\subset\; G \times H'$$

together with a mapping $\mathcal{L} \longrightarrow \mathcal{L}'$ defined by $(x, x^\star) \mapsto (x, \rho(x^\star))$. This is a group homomorphism, and surjective. Since $(x, x^\star) \mapsto (0, 0) \in \mathcal{L}'$ is only possible for $x = 0$, we see that

also $\rho(x^\star) = 0$ in this case. Consequently, the kernel of this homomorphism is $\{(0,0)\}$, and $\mathcal{L} \simeq \mathcal{L}'$.

Consider the diagram

$$\begin{array}{ccc} G \times H & \xrightarrow{\mathrm{id}\times\rho} & G \times H' \\ {\scriptstyle \mathrm{nat}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{nat}} \\ (G\times H)/\mathcal{L} & \longrightarrow & (G \times H')/\mathcal{L}' \end{array}$$

where the horizontal arrow in the lower line exists in an obvious way, because $(x, x^\star) \in \mathcal{L}$ is mapped to $(x, \rho(x^\star)) \in \mathcal{L}'$. Since $\mathcal{L}$ is a closed subgroup of $G \times H$, quotient is Hausdorff. Moreover, the natural mapping $G \times H \longrightarrow (G \times H)/\mathcal{L}$ is an open map, and we get that $(G \times H)/\mathcal{L} \longrightarrow (G \times H')/\mathcal{L}'$ is continuous.

Consequently, $(G \times H')/\mathcal{L}'$ is compact, whence $\mathcal{L}'$ is co-compact. Consider now a compact neighbourhood $U \times \rho(V)$ of 0 in $G \times H'$, where V is a compact neighbourhood of 0 in H and U is a compact neighbourhood of 0 in G. Then,

$$\mathcal{L}' \cap (U \times \rho(V)) \; = \; \{(x, \rho(x^\star)) : x \in U,\, \rho(x^\star) \in V\} \; = \; \{(x, x^\star) : x \in U,\, x^\star \in V + I\}\,.$$

Since $U \times (V + I)$ is compact and $\mathcal{L}$ is a lattice, the set $\{(x, x^\star) : x \in U, x^\star \in V + I\}$ is *finite*, and contains $(0, 0^\star) = (0, 0)$. Consequently, $\mathcal{L}' \cap \bigl(U \times \rho(V)\bigr)$ is finite, too, with $(0,0) \in \mathcal{L}' \cap \bigl(U \times \rho(V)\bigr)$. Consequently, $(0,0)$ is isolated and $\mathcal{L}'$ is (uniformly) discrete.

This shows that $(G, H', \mathcal{L}')$ is another CPS, with all the properties required, for which we now need a window. To this end, define $W' = \rho(W)$. We note that W' is compact since W is. Moreover, also W is the complete preimage of $\rho(W)$ since $W = I + W$. As ρ is an open map, $\rho(W^\circ)$ is open and we have $W'^\circ = \rho(W^\circ)$ and $W' = \overline{W'^\circ} \neq \varnothing$.

Note that $\rho \colon H \longrightarrow H'$ induces a mapping from the Haar measure θ_H to a Haar measure $\theta_{H'}$, with

$$\theta_{H'}(\partial W') \; = \; \theta_{H'}(W' \setminus W'^\circ) \; = \; \theta_H\bigl((W + I) \setminus (W^\circ + I)\bigr) \; = \; \theta_H(W \setminus W^\circ) \; = \; 0\,.$$

Finally, if $\curlywedge(W^\circ) \subset \varLambda \subset \curlywedge(W)$ for some $\varLambda \subset G$ in the original CPS, then

$$\curlywedge(W'^\circ) \; \subset \; \varLambda \; \subset \; \curlywedge(W')$$

in the new CPS. This is easy to check because $x \in \curlywedge(W'^\circ) \iff \rho(x^\star) \in W'^\circ \iff x^\star \in W^\circ$ and similarly for the remaining details. This completes the argument. □

9.2. **Torus parametrizations for model sets.** Assume that we are given an IMS $\varLambda$ with respect to the CPS $(G, H, \mathcal{L})$ with a window W. According to the previous paragraph, we may assume that the CPS is irredundant, and then, by Proposition 7, that we have a torus parametrization of the local hull into the compact group $\mathbb{T}$ of this scheme. We shall now convert this to a torus map into $\mathbb{A}(\varLambda)$.

Lemma 8. *Let $(G, H, \mathcal{L})$ be a CPS and let $W \subset H$ be a compact, non-empty and irredundant window, with $W = \overline{W^\circ}$ and $\theta_H(\partial W) = 0$. Then, the following holds.*

(a) *$\theta_H((W - h_n) \triangle W) \longrightarrow 0$, whenever $\{h_n\}$ is a net in H with $h_n \to 0 \in H$.*
(b) *If $h \in H$ satisfies $\theta_H((W - h) \triangle W) = 0$, then $h = 0$.*

Proof. (a) Denote the canonical representation of H on $L^1(H,\theta_H)$ by τ^H, i.e., $\tau_h^H f(x) = f(-h+x)$. It is well known that this representation is strongly continuous [32]. This means that $\tau_h^H f \longrightarrow f$ for $h \to 0$ and all $f \in L^1(H,\theta_H)$. Now, let 1_W be the characteristic function of the compact set $W \subset H$. Then, 1_W belongs to $L^1(H,\theta_H)$ and therefore

$$\theta_H((W-h) \triangle W) \;=\; \|\tau_h^H 1_W - 1_W\|_{L^1} \;\longrightarrow\; 0\,, \quad \text{as } h \to 0\,.$$

This proves (a).

(b) Let $h \in H$ be given with $\theta_H((W-h) \triangle W) = 0$. As the window is translationally fixed only by $0 \in H$, it suffices to show $W - h = W$, i.e., $(W-h) \triangle W = \varnothing$. Assume the contrary. Then, $(W-h) \triangle W$ actually contains an open set because $W = \overline{W^\circ}$. This implies $\theta_H((W-h) \triangle W) > 0$ and we arrive at a contradiction. $\square$

Proposition 13. *Let $(G,H,\mathcal{L})$ be a CPS with associated torus $\mathbb{T}$. Let $W \subset H$ be a compact, non-empty and irredundant window, with $W = \overline{W^\circ}$ and $\theta_H(\partial W) = 0$. Let $\varLambda \subset G$ be an arbitrary IMS, i.e., $\curlywedge(W^\circ) \subset \varLambda \subset \curlywedge(W)$. Then, selecting a neighbourhood U so that $\varLambda \in \mathcal{D}_U$, the mapping $j : \mathbb{T} \longrightarrow \mathcal{D}_U^{\equiv}$, $(x,h) + \mathcal{L} \mapsto \beta(x + \curlywedge(W-h))$, is continuous and injective. In particular, j gives an isomorphism between $\mathbb{T}$ and $\mathbb{A}(\varLambda)$.*

Proof. We first show continuity of j. Let $\{\xi_\iota\}$ be a net in $\mathbb{T}$ with $\xi_\iota \longrightarrow \xi \in \mathbb{T}$. Then, without loss of generality, we may assume that $\xi = (x,h) + \mathcal{L}$, $\xi_\iota = (x_\iota, h_\iota) + \mathcal{L}$ and $x_\iota \longrightarrow x$, $h_\iota \longrightarrow h$. As the topology of $\mathcal{D}_U^{\equiv}$ allows for small translations, it suffices to show that $\beta(\curlywedge(W-h_\iota)) \longrightarrow \beta(\curlywedge(W-h))$. By part (a) of Lemma 8, we infer

$$\theta_H((W-h_\iota) \triangle (W-h)) \;\longrightarrow\; 0\,. \tag{27}$$

Therefore,

$$\begin{aligned} d\big(\beta(\curlywedge(W-h_\iota)), \beta(\curlywedge(W-h))\big) \quad &= \quad \mathrm{dens}\big(\curlywedge(W-h_\iota) \triangle \curlywedge(W-h)\big) \\ &= \quad \mathrm{dens}\big(\curlywedge((W-h_\iota) \triangle (W-h))\big) \\ &\overset{\text{by Thm. 4}}{=} \quad \mathrm{dens}(\mathcal{L})\, \theta_H((W-h_\iota) \triangle (W-h)) \\ &\xrightarrow{\text{by (27)}} \quad 0. \end{aligned}$$

This proves the continuity statement.

We now show injectivity. To do so, let (x,h) and (x',h') in $G \times H$ be given with $\beta(x + \curlywedge(W-h)) = \beta(x' + \curlywedge(W-h'))$. This implies

$$\beta(\curlywedge(z^\star + k + W')) \;=\; \beta(\curlywedge(W'))\,,$$

where $z = x - x'$, $k = h' - h$ and $W' = -h' + W$. By Proposition7 and Theorem 4, we then obtain

$$\begin{aligned} \mathrm{dens}(\mathcal{L})\, \theta_H\big((z^\star + k + W') \triangle \curlywedge(W')\big) \;&=\; \mathrm{dens}\big(\curlywedge(z^\star + k + W' \triangle \curlywedge(W')\big) \\ &=\; d\big(\beta(\curlywedge(z^\star + k + W')), \beta(\curlywedge(W'))\big) \;=\; 0. \end{aligned}$$

By part (b) of Lemma 8, this gives $0 = k + z^\star$ or, put differently, $(x,h) + \mathcal{L} = (x',h') + \mathcal{L}$. This proves injectivity.

The inverse of a continuous injective map on a compact space is continuous as well.

So far, we know that j is a continuous bijective map from $\mathbb{T}$ onto $j(\mathbb{T})$. By continuity of j and minimality of the action of G on $\mathbb{T}$, $j(\mathbb{T})$ is just $\mathbb{A}(\curlywedge(W))$. By uniform distribution, $\beta(\curlywedge(W)) = \beta(\varLambda)$ and $\mathbb{A}(\curlywedge(W)) = \mathbb{A}(\varLambda)$ follows. This proves the last statement. □

Remark 5. It is known from [28] that $\mathbb{T}$ and $\mathbb{A}(\varLambda)$ are isomorphic for regular model sets, and our proof is in some sense a variant of the proof in [28]. However, our result here is more explicit in that β is shown to establish this isomorphism, and this will allow us to clarify the relationship between β and $\beta_{\mathbb{A}}$.

Theorem 9. *Let $(G, H, \mathcal{L})$ be a CPS, and let a non-empty window $W \subset H$ with $W = \overline{W^\circ}$ and $\theta_H(\partial W) = 0$ be given. If $\varLambda \subset G$ satisfies $t + \curlywedge(W^\circ) \subset \varLambda \subset t + \curlywedge(W)$ for some $t \in G$, then the canonical mapping $\beta\colon\ \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$ is continuous and one-to-one almost everywhere.*

Proof. By Lemma 7, we may assume, without loss of generality, that the cut and project scheme is irredundant. Assume furthermore that $t = 0$. Proposition 7 then gives a torus parametrization $\beta_{\mathbb{T}}\colon\ \mathbb{X}(\varLambda) \longrightarrow \mathbb{T}$ with

$$\beta_{\mathbb{T}}(\varLambda') = (x, h) + \mathcal{L} \iff x + \curlywedge(-h + W^\circ) \subset \varLambda' \subset x + \curlywedge(-h + W).$$

By $\theta_H(\partial W) = 0$ and Theorem 4, $\beta(\varLambda') = \beta(x + \curlywedge(-h + W))$ whenever $x + \curlywedge(-h + W^\circ) \subset \varLambda' \subset x + \curlywedge(-h+W)$. Thus, Proposition 13 shows that $\beta = j \circ \beta_{\mathbb{T}}$ with a continuous injective j. Thus, β is a continuous torus parametrization. It remains to show that it is one-to-one almost everywhere. But this follows from Theorem 5. □

Remark 6. Proposition 13 and the considerations in the proof of the previous theorem effectively show that the maps $\beta_{\mathbb{A}}\colon\ \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}$ and $\beta_{\mathbb{T}}\colon\ \mathbb{X}(\varLambda) \longrightarrow \mathbb{T}$, yielding a description of $\varLambda$ as a regular model set in the previous sections, can be identified with the canonical map β.

9.3. **The end of Theorem 1.**

Proof. Let $\varLambda$ be a regular model set. By Lemma 7, we may assume that its CPS is irredundant and that we are in the situation of Theorem 9. This provides us with an almost everywhere one-to-one continuous mapping of $\mathbb{X}(\varLambda)$ onto $\mathbb{A}(\varLambda)$. Theorem 6 shows that $\mathbb{X}(\varLambda)$ is uniquely ergodic, and, since $\varLambda$ is repetitive, $\mathbb{X}(\varLambda)$ is also minimal. By Theorem 7, we obtain a pure point dynamical spectrum with continuous eigenfunctions that separate almost all points of $\mathbb{X}(\varLambda)$. □

10. The crystallographic case

The aim of this section is to give a proof of the following result on the characterization of fully periodic Delone sets.

Theorem 10. *Let G be an LCA group and $\varLambda$ a uniformly discrete subset of G. Then, the following assertions are equivalent.*

(i) *$\varLambda$ is crystallographic.*
(ii) *$\varLambda$ is Meyer and the map $\beta\colon\ \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$ is continuous and injective.*

(iii) *All of the following conditions hold:*
  (1) *All elements of* $\mathbb{X}(\varLambda)$ *are Meyer sets.*
  (2) $(\mathbb{X}(\varLambda), G)$ *is uniquely ergodic.*
  (3) $(\mathbb{X}(\varLambda), G)$ *has pure point dynamical spectrum with continuous eigenfunctions.*
  (4) *The eigenfunctions separate all points of* $\mathbb{X}(\varLambda)$.

In this case, $(\mathbb{X}(\varLambda), G)$ *is also minimal, hence strictly ergodic.*

Recall that $\varLambda$ is called *crystallographic* (or fully periodic) if its periods

$$\mathrm{per}(\varLambda) \;:=\; \{t \in G : t + \varLambda = \varLambda\}$$

form a lattice, i.e., a co-compact discrete subgroup of G.

We start by proving the equivalence of claims (i) and (ii) of Theorem 10.

Lemma 9. *The Delone set* $\varLambda \subset G$ *is crystallographic if and only if* $\varLambda$ *is Meyer and the mapping* $\beta \colon \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$ *is continuous and injective.*

Proof. The 'only if' part is easy: If the Delone set $\varLambda$ is crystallographic, with lattice of periods $P = \mathrm{per}(\varLambda)$, it is of the form $\varLambda = F + P$, where F must be a finite set due to the uniform discreteness of $\varLambda$. From $P - P = P$, we have $\varLambda - \varLambda = (F - F) + P$. This is still uniformly discrete, because $F - F$ is still finite, and $\varLambda$ is Meyer.

Let $\mathbb{T} = G/\mathrm{per}(\varLambda)$ be the compact quotient of G by the set of periods. Then, there is a natural isomorphism $\mathbb{T} \longrightarrow \mathbb{X}(\varLambda)$. This easily yields the statement about β.

We now prove the 'if' statement: As β is continuous, $\mathbb{A}(\varLambda)$ is compact and we have pure point diffraction. In particular, the ε-almost periods P_ε are relatively dense.

Obviously, $\mathrm{per}(\varLambda)$ is a subgroup of G. As $\varLambda$ is a Meyer set, $\mathrm{per}(\varLambda)$, being a subset of $\varLambda - \varLambda$, is uniformly discrete. Therefore, it suffices to show that the set of periods is relatively dense (which implies that the quotient $G/\mathrm{per}(\varLambda)$ is compact). We shall show that the set of periods contains the set of P_ε of ε-almost periods for a suitable $\varepsilon > 0$. As P_ε is relatively dense, the desired statement follows. Here are the details:

As $\mathbb{A}(\varLambda)$ is compact and β is continuous and injective by assumption, it has a continuous inverse

$$\alpha \colon \mathbb{A}(\varLambda) \longrightarrow \mathbb{X}(\varLambda).$$

As $\mathbb{A}(\varLambda)$ is compact, α is even uniformly continuous. Thus, for every compact $C \subset G$ and open $V \subset G$, there exists an $\varepsilon > 0$ such that

$$(\alpha(\xi) + t, \alpha(\xi)) = (\alpha(\xi + t), \alpha(\xi)) \in U_{\mathrm{LT}}(C, V) \tag{28}$$

for all $\xi \in \mathbb{A}$ and all $t \in P_\varepsilon$, where use the addition of t for the translation action on both spaces for simplicity. Here, of course,

$$U_{\mathrm{LT}}(C, V) \;=\; \{(\varGamma, \varGamma') : (\varGamma + v) \cap C = \varGamma' \cap C \text{ for a suitable } v \in V\}.$$

Now, choose an open neighbourhood V of 0 in G according to Fact 4, meaning that we have $V \cap ((\varLambda - \varLambda) + (\varLambda - \varLambda)) = \{0\}$, and a compact set $C \subset G$ such that, for all $t \in G$, $(t + C) \cap \varGamma \neq \varnothing$.

Choose $\varepsilon > 0$ so that (28) holds. As α is onto, we infer that, for every $\varGamma \in \mathbb{X}(\varLambda)$ and every $t \in P_\varepsilon$,

$$(\varGamma + t, \varGamma) \in U_{\mathrm{LT}}(C, V).$$

Fact 4 then implies

$$(\varGamma + t) \cap C = \varGamma \cap C$$

for every $\varGamma \in \mathbb{X}(\varLambda)$ and $t \in P_\varepsilon$. As $\varGamma$ is arbitrary, we can replace $\varGamma$ by $s + \varGamma$ with $s \in G$ arbitrary. Thus, we end up with

$$(\varGamma + s + t) \cap C = (\varGamma + s) \cap C$$

for every $s \in G$. As $C \neq \varnothing$, this shows that every $t \in P_\varepsilon$ is a period of $\varGamma$. □

We now show equivalence of (ii) and (iii).

Lemma 10. *A set $\varLambda \subset G$ is Meyer and $\beta \colon \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$ is continuous and injective if and only if the following 4 conditions hold.*

1. *All elements of $\mathbb{X}(\varLambda)$ are Meyer sets;*
2. *$(\mathbb{X}(\varLambda), G)$ is uniquely ergodic;*
3. *$(\mathbb{X}(\varLambda), G)$ has pure point dynamical spectrum with continuous eigenfunctions;*
4. *The eigenfunctions separate all points of $\mathbb{X}(\varLambda)$.*

Proof. We first show the "only if" part: The validity of (1) is clear as $\varLambda$ is Meyer. By assumption on β, $\mathbb{X}(\varLambda)$ is isomorphic to the group $\mathbb{A}(\varLambda)$. Now, $(\mathbb{A}(\varLambda), G)$ is uniquely ergodic (as the action of G is minimal and $\mathbb{A}(\varLambda)$ is a group) and it has pure point dynamical spectrum with continuous eigenfunctions given by the characters. As, by assumption on β, the dynamical system $(\mathbb{X}(\varLambda), G)$ is topologically conjugate to $(\mathbb{A}(\varLambda), G)$, the assertions (2), (3) and (4) follow.

We now prove the "if" part: We can apply Theorem 7 to obtain a continuous mapping $\beta_{\mathbb{A}}:$ $\mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$ as we have pure point dynamical spectrum with continuous eigenfunctions. Moreover, again by Theorem 7, the map is actually injective because the eigenfunctions separate *all* points.

It remains to show that $\beta_{\mathbb{A}}$ agrees with β. We first note that $(\mathbb{X}(\varLambda), G)$ is minimal, because $(\mathbb{A}(\varLambda), G)$ is minimal and $\beta_{\mathbb{A}}$ is injective. By injectivity of $\beta_{\mathbb{A}}$ and Theorem 3, $(\mathbb{X}(\varLambda), G)$ is associated with a repetitive model set. In fact, the model set is regular by Theorem 5. Thus, by Theorem 9, the map β is continuous. Consequently, β and $\beta_{\mathbb{A}}$ are continuous G-maps from $\mathbb{X}(\varLambda)$ into $\mathcal{D}_U^{\equiv}$, for suitable U, which agree on $\varLambda$. Therefore, they agree everywhere. □

Appendix: Meyer sets in locally compact Abelian groups

Let G be a locally compact Abelian (LCA) group and let $\varLambda$ be a Delone subset of G. We wish to compare the following two properties that $\varLambda$ may have.

M1 $\varLambda - \varLambda \subset \varLambda + F$ for some finite set F;
M2 $\varLambda - \varLambda$ is uniformly discrete.

Often these properties are taken as (equivalent) characterizations of Meyer sets, which they are for groups of the form $\mathbb{R}^d$. In fact, it is easy to see that **M1** always implies **M2**. The reverse implication for $\mathbb{R}^d$ was proved by Lagarias [17]. Here, we prove it for all compactly generated LCA groups. The proof goes in two steps. First, we prove that the group generated by $\varLambda$ is finitely generated. In fact, this is equivalent to saying that G is compactly generated. After that, we can basically follow Lagarias' proof in the more general setting.

An apparently weaker concept than uniform discreteness is weak uniform discreteness:

Definition 6. $S \subset G$ is *weakly uniformly discrete* if for each compact subset K of G and for all $a \in G$, $\mathrm{card}(S \cap (a + K))$ is bounded by a constant that depends only on K (not on a).

Remarkably, as we shall see, for a Delone subset $\varLambda$ of a compactly generated group G, the difference set $\varDelta := \varLambda - \varLambda$ is uniformly discrete if and only if it is weakly uniformly discrete.

Proposition 14. *If a Delone set $\varLambda$ satisfies* **M1***, it also satisfies* **M2***.*

Proof. Since $\varLambda$ is a Delone set, it is locally finite, i.e., $\varLambda \cap K$ is a finite set (or empty), for any compact set $K \subset G$. To establish the uniform discreteness of $\varDelta$, it is sufficient to show that 0 is an isolated point of $\varDelta - \varDelta$. Using **M1** twice, one has

$$0 \;\in\; \varDelta - \varDelta \;\subset\; (\varLambda + F) - (\varLambda + F) \;=\; \varDelta + (F - F) \;\subset\; \varLambda + F'$$

where $F' = F + F - F$ is still a finite set. Consequently, $\varLambda + F'$ is locally finite, and 0 must be an isolated point of it, hence also of $\varDelta - \varDelta$. This gives **M2**. □

Proposition 15. *Let G be an LCA group and let $\varLambda$ be relatively dense in G. Suppose that $\langle \varLambda \rangle$ (the subgroup of G generated by $\varLambda$) is finitely generated. Then, G is compactly generated.*

Proof. Let F be a finite set that generates $\langle \varLambda \rangle$. As $\varLambda$ is relatively dense, there is a compact set $K \subset G$ with $G = \varLambda + K$. Then, $F \cup K$ is compact and generates G. □

Lemma 11. *Let G be an LCA group of the form $G' \times T$ where T is a compact group. Then, the projection mapping $G \longrightarrow G'$ defined by this splitting maps locally finite sets to locally finite sets.*

Proof. Suppose that $S \subset G$ is locally finite, but its projection P' is not. Then, there exists a compact set $K \subset G'$ with $P' \cap K$ infinite and we have $S \cap (K \times T)$ infinite, too. This is a contradiction because $K \times T$ is compact. □

Proposition 16. *Let G be compactly generated. Let $\varLambda \subset G$ be relatively dense and suppose that $\varDelta = \varLambda - \varLambda$ is locally finite. Then, $\langle \varLambda \rangle$ is finitely generated.*

Proof. By the structure theorem for compactly generated LCA groups [13, Thm. 9.8], G is isomorphic to $\mathbb{R}^m \times \mathbb{Z}^n \times T$, where T is compact. We identify G with this group and so can also view it as a subgroup of $\mathbb{R}^m \times \mathbb{R}^n \times T =: G' \times T$. We shall use $(')$ to indicate the projection map of $G' \times T$ onto G'. By Lemma 11, $\varDelta' \subset G'$ is locally finite. Also, $\varLambda$ is locally finite, due to the corresponding property of $\varDelta$.

Select a compact set $C \subset G$ so that $\varLambda + C = G$. Since the projection of C into G' is compact, we can find $R > 0$ so that $\varLambda + (B_R \times T) \supset \mathbb{R}^m \times \mathbb{Z}^n \times T$, where B_R is the open ball of radius R around 0 in $\mathbb{R}^m \times \mathbb{R}^n$. Increasing R if necessary, we may assume

$$\varLambda + (B_R \times T) \;=\; \mathbb{R}^m \times \mathbb{R}^n \times T.$$

Consider $F := (\varLambda \cup (\varLambda - \varLambda)) \cap (B_{2R} \times T)$, which is finite. It is plain that $F \subset \langle \varLambda \rangle$. We show that $\langle \varLambda \rangle = \langle F \rangle$. In fact, $\varLambda$ is contained in the semigroup generated by F.

Let $\lambda \in \varLambda$. If $\lambda \in B_{2R} \times T$, then $\lambda \in F$, so suppose $\lambda \notin B_{2R} \times T$. We need to get closer to 0, using a point of F. To this end, let $B_R(u')$ be the open ball of radius R around u' in G', where u' is taken to be the unique point which is at distance R from λ' on the line segment $[0, \lambda']$ in $\mathbb{R}^m \times \mathbb{R}^n$ joining 0 to λ'. Thus, $\lambda' \in \overline{B_R(u')} \setminus B_R(u') = \partial B_R(u')$. Let $u := (u', 0) \in G' \times T$. By our choice of R, we can write $u = \mu_1 + b$, where $\mu_1 \in \varLambda, b \in B_R \times T$, so $\mu_1' = u' - b' \in B_R(u')$.

We have (i) $\mu_1 \in \varLambda$, (ii) $|\mu_1'| < |\lambda'|$, (iii) $|\lambda' - \mu_1'| < 2R$, where $|.|$ is the standard Euclidean norm in $\mathbb{R}^{m+n}$. Also, $\lambda - \mu_1 \in (B_{2R} \times T) \cap \varDelta \subset F$ and so we have

$$\lambda = f_1 + \mu_1, \quad \text{with} \quad f_1 \in F,\, \mu_1 \in \varLambda,\, |\mu_1'| < |\lambda'|.$$

We now continue inductively, getting

$$\lambda \;=\; f_1 + \ldots + f_k + \mu_k$$

where $f_1, \ldots, f_k \in F$, $\mu_k \in \varLambda$ and $|\mu_k'| < |\mu_{k-1}'| < \cdots < |\lambda'|$, until $|\mu_k'| < 2R$. This must happen for some k since $\varLambda'$ is locally finite and $\varLambda' \cap \overline{B_{\lambda'}(0)}$ is finite. Then, $\mu_k \in F$ and we have shown that $\lambda \in \langle F \rangle$. □

Theorem 11. *Let G be a compactly generated LCA group. Suppose that $\varLambda$ is a relatively dense subset of G and that $\varLambda - \varLambda$ is weakly uniformly discrete. Then, $\varLambda$ satisfies* **M1**.

The proof of this result is really not different from that given in [17]. The only things to notice are that the full strength of uniform discreteness of $\varLambda - \varLambda$ is not required and that we are no longer confined to real spaces.

Proof. We may assume that $0 \in \varLambda$, translating $\varLambda$ if necessary. Let $L := \langle \varLambda \rangle$ which is finitely generated by Proposition 16: say $\langle \varLambda \rangle = \langle e_1, \ldots, e_s \rangle$. For all $x \in L$, $x = \sum_{i=1}^{s} a_i e_i$, $a_i \in \mathbb{Z}$, though not necessarily uniquely.

Define $\|x\| = \min\{\sum |a_i| : x = \sum a_i e_i\}$. This defines a norm on L (where the proof of the triangle inequality requires a short calculation). For each $N \in \mathbb{N}$, define $F(N) := \{x \in L : \|x\| \leq N\}$, which is clearly a finite set.

Let $K \subset G$ be a symmetric compact neighbourhood of 0 so that, for every $u \in G$, we have $(u + K) \cap \varLambda \neq \varnothing$, and also that G is generated as a group by K. The goal is now to show the existence of finitely many "stepping stones", forming a set F, such that any difference $x - y$ of points in $\varLambda$ lies in $\varLambda + F$.

To this end, let

$$\begin{aligned} M &:= \max\{\operatorname{card}((u + 2K) \cap (\varLambda - \varLambda)) : u \in G\}, \\ m &:= \max\{\|u\| : u \in (\varLambda - \varLambda) \cap (K + K - K)\}. \end{aligned}$$

The former exists on the basis of the weak uniform discreteness of $\varLambda - \varLambda$.

Let $x, y \in \varLambda$ and let $v := y - x$. Now, $x \in \ell K := K + \ldots + K$ (ℓ summands) for some ℓ, and we may write

$$x = k_1 + \ldots + k_\ell\,, \quad \text{for some } k_i \in K.$$

Let $x_0 := x, x_1 := x - k_1, x_2 := x - k_1 - k_2, \ldots, x_\ell = 0$ and define the parallel sequence $y_i := x_i + v$, $0 \leq i \leq \ell$. Then, $y_i - y_{i+1} = x_i - x_{i+1} = k_{i+1} \in K$, for all $0 \leq i \leq \ell - 1$.

Choose $p_i, q_i \in \varLambda$ with $p_i - x_i, q_i - y_i \in K$ with the special choices $p_0 = x, q_0 = y, p_\ell = 0 = x_\ell$. Note that $q_0 = y = x + v = x_0 + v = y_0$, so in particular $q_0 - y_0 = 0 \in K$. Then, for each $i \in \{0, 1, \dots, \ell\}$, one has $q_i - p_i - v = q_i - y_i + x_i - p_i \in 2K$. Thus,

$$V \;:=\; \{q_i - p_i : i = 0, \dots, \ell\} \subset \{(v + 2K) \cap (\varLambda - \varLambda)\},$$

so $\mathrm{card}(V) \leq M$.

Similarly,

$$p_i - p_{i+1} \;=\; (p_i - x_i) + (x_i - x_{i+1}) + (x_{i+1} - p_{i+1}) \;\subset\; (K + K - K) \cap (\varLambda - \varLambda)$$

so $\|p_i - p_{i+1}\| \leq m$.

In the same way, $\|q_i - q_{i+1}\| \leq m$, so

$$\|q_i - p_i - (q_{i+1} - p_{i+1})\| \;\leq\; 2m.$$

Along with the bound on the cardinality of V, this gives rise to

$$\|u - u'\| \;\leq\; 2mM$$

for all $u, u' \in V$.

Now, $v = q_0 - p_0 = y - x \in V$ and $q_\ell = q_\ell - p_\ell \in V$, so $\|v - q_\ell\| \leq 2mM$. Since $v, q_\ell \in L$, $v - q_\ell \in F(2mM)$ and

$$y - x \;=\; v \in q_\ell + F(2mM) \;\subset\; \varLambda + F(2mM).$$

Since $x, y \in \varLambda$ were arbitrary, $\varLambda - \varLambda \subset \varLambda + F(2mM)$. □

Corollary 1. *Let G be a compactly generated LCA group. Let $\varLambda \subset G$ be a Delone set. Then, $\varLambda - \varLambda$ is uniformly discrete if and only if it is weakly uniformly discrete.*

Proof. Use Theorems 11 and 14. □

Acknowledgements. It is our pleasure to thank Jeong-Yup Lee, Bernd Sing and Nicolae Strungaru for discussions and comments on the manuscript. This work was supported by the German Research Council (DFG), within the CRC 701, and the Natural Sciences and Engineering Council of Canada (NSERC).

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